Pareto optima and exchange rates under risk neutrality: A note

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A note∗

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Abstract
In this note, we present a wealth model of a two-country economy where financial assets and goods are traded. We consider the case where the agents are risk neutral, a very common assumption in finance in order to have explicit solutions for prices, and in particular in international finance for exchange rates using the Pareto optima. We show that the Pareto optima on the international good and capital markets are found to coincide with the net trade allocations. More notably, under a no-arbitrage condition on the international capital market, we can define an exchange rates system for which Purchasing Power Parity (PPP) holds. And if any Pareto optimum for the international capital market and its associated commodity prices clear the trade balance then the Uncovered Interest Rate Parity (UIRP) for the international capital market holds with this exchange rates system. When the international financial market are complete, this condition is also sufficient for the trade balance with any Pareto allocation and its associated commodity prices. In this case, PPP on the international good market and UIRP for the international capital market are equivalent conditions. We show through an example of risk-neutral economy where a no-arbitrage condition for assets and PPP hold, but UIRP fails, that the only individually rational Pareto allocation clearing the international good market is no-trade. Finally, we recover Dumas [3] under risk aversion in a simplified two-country economy with a single financial asset and no exchange of commodities and we prove that the only possible equilibrium (with transfers) is no-trade.

Keywords: international asset pricing, returns on securities, exchange rates, no-arbitrage condition.

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1 Introduction

In this note, we consider a pure exchange economy with financial assets and international good trade. Under the assumption of risk neutrality, we determine the exchange rates explicitly by using Pareto optima. Most of the papers on finance rest on the assumption of risk neutrality because it gives the possibility to have explicit solutions for the prices at the equilibrium. In international finance, this assumption permits to get an explicit solution for the exchange rates. Our main result is the following. If no arbitrage condition holds, and if any Pareto optimum for the international capital market and its associated commodity prices clear also the international good market, then the Uncovered Interest Rate Parity (UIRP) for the international capital market is satisfied. When the international financial markets are complete, the assets are not redundant, and no arbitrage condition holds, then if UIRP holds then the Purchasing Power Parity (PPP) also holds. Moreover, any Pareto optimum and its associated prices will clear the trade balance.

When the UIRP fails, under risk neutrality, we can again endogenize the exchange rates and we show through a simple example that, unfortunately, the set of Pareto optima may consist of no-trade allocations.

We revisit a two-period financial model by Hart [4] to take into account the international trade. In the first period, agents trade financial assets to diversify their portfolios and maximize a linear utility function (risk neutrality). In the second period, they exchange goods spending their initial endowments and the gains from financial investments. They are allowed to exchange goods across the borders, contrarily to Dumas [3]. Security returns and goods are valued in domestic currencies.

In the financial literature, few authors have been able to endogenize the exchange rates. Among them, Dumas [3] looks like a pioneer. He solves a symmetric Pareto optimum for a two-country model with one consumption good, one capital good and obtains the behavior of capital transfers between the countries (the balance of trade), consumption, investment, the real exchange rate (defined as the relative price of capital goods between two countries), and the real interest-rate differential. In this two-country economy, agents consume only the domestic goods but exchange financial assets and are risk averse. However, at page 172, he mentions that, as the agents approach the risk neutral behavior, the economy he considers tends to a no-trade economy and that the PPP almost holds.

In Dumas’ contribution, UIRP seems to be equivalent to PPP. In the spirit of Dumas [3], we focus on the set of Pareto allocations but only under the assumption of risk neutrality. We calculate the exchange rates and find that they depend on the agents’ beliefs and on financial returns. Two main differences between Dumas’ model and ours: (1) we have many financial assets and (2) agents have heterogeneous beliefs. We prove (see Remark 14), that, in the case of one capital good, UIRP turns out to be equivalent to PPP as alluded by Dumas at page 174.

The case of many financial assets strongly differs from the single asset case.
In particular, we exhibit an example where the set of Pareto optima compatible with the explicit exchange rates is the set of no-trade allocations. In this respect, we conclude that the assumption of risk-neutral agents (in order to endogenize the exchange rates through the Pareto optima with risk-neutral agents) has no interest because the solution consists of no trade allocations. We prove what Dumas [3] obtains by simulations.

We conclude the paper with a section on risk aversion to found Dumas [3] on a theoretical ground and derive a surprising result: in an oversimplified economy with a single financial asset and no commodity exchange as in Dumas [3] the only possible equilibrium (with transfers) is no (asset) trade.

Our note is organized in ten sections. Notations and fundamentals are given in Sections 2 to 5. The no-arbitrage condition in the financial market is considered in Section 6. Section 7 bridges risk neutrality, Pareto optimality and the exchange rates, and states our main result. Section 8 provides an example of a no-trade economy. Section 9 bridges Dumas [3] and our model. Section 10 concludes.

2 Model

We focus on a pure exchange economy where financial assets and goods are traded in international markets. More precisely, we consider a two-period exchange economy with many countries. Financial assets are traded in the first period and goods are consumed in the second. The representative agent of country \( i \in \{0, \ldots, I\} \) purchases \( K \) financial assets in period 0 to smooth consumption in period 1 across \( S \) states of nature. Nature provides an endowment in period 1.

3 Notations

Let us introduce a compact notation for asset prices and quantities on the financial side of the economy.

- \( q \equiv (q_1, \ldots, q_K) \) is a row of financial asset prices where \( q_k \) denotes the price of financial asset \( k \).
- \( x \equiv (x^i_k) \) is the \( K \times (1 + I) \) matrix of portfolios where \( x^i_k \) denotes the amount of financial asset \( k \) in the portfolio of agent \( i \). Column \( x^i \equiv (x^i_1, \ldots, x^i_K)^T \) is the portfolio of agent \( i \).
- \( R^i \equiv (R^i_{sk}) \) is the \( S \times K \) matrix of returns where \( R^i_{sk} \geq 0 \) denotes the return\(^1\) on asset \( k \) in the state of nature \( s \). \( R^i_s \) is the \( s \)th row of the matrix. Returns \( R^i_{sk} \) are valued in the currency of country \( i \).

\(^1\)The return is the value of one unit of security in the second period including the dividend. Agents form beliefs about the future and associate each return with the probability of its state of nature.
Let us provide now a compact notation for beliefs, prices and quantities on
the real side of the economy.

\( \pi \equiv (\pi_s^i) \) is the \((1+I) \times S\) matrix of beliefs where \( \pi_s^i \)
denotes the belief of agent \( i \) about the occurrence of state \( s \). The individual row of beliefs \( \pi^i \equiv (\pi_1^i, \ldots, \pi_S^i) \) lies in the \( S \)-unit simplex.

\( p \equiv (p_s^i) \) is the \((1+I) \times S\) matrix of good prices where \( p_s^i \)
denotes the price of the good in country \( i \) in the state of nature \( s \). \( p^i \equiv (p_1^i, \ldots, p_S^i) \) is the \( i \)th row of the matrix.

\( \tau \equiv (\tau_s^i) \) is the \((1+I) \times S\) matrix of exchange rates where \( \tau_s^i \)
denotes the exchange rate between currencies of country 0 and country \( i \) in the state of
nature \( s \). \( \tau^i \equiv (\tau_1^i, \ldots, \tau_S^i) \) is the \( i \)th row of the matrix. The first row is a vector of units: \( \tau_s^0 = 1 \) for any \( s \).

\( w \equiv (w_s^i) \) is the \( S \times (1+I) \) wealth matrix where \( w_s^i \) denotes the wealth enjoyed by agent \( i \) in the state of nature \( s \). \( w^i \equiv (w_1^i, \ldots, w_S^i)^T \) is the wealth column of agent \( i \). The amount \( w_s^i \) is valued in the currency of country \( i \) and the utility function of any agent depends on her wealth:
\( u_i = u_i(w_s^i) \).

\( e \equiv (e_s^i) \) is the \( S \times (1+I) \) matrix of endowments where \( e_s^i \) denotes the endowment nature provides to agent \( i \) in the state \( s \). \( e^i \equiv (e_1^i, \ldots, e_S^i)^T \) is the endowment column of agent \( i \). The endowment \( e_s^i \) is valued in the currency of country \( i \).

Notice that prices and beliefs \( q, R_s^i, \pi^i, p^i, \pi^i \) are rows, while quantities \( x^i, w^i, e^i \) are columns.

The individual consumption value is given by \( p^i w^i \). Since \( w_s^i \) is valued in the currency of country \( i \), we interpret \( p_s^i \) as the inverse of an ordinary price and \( p_s^i w_s^i \) as the physical value of \( w_s^i \). The physical values can be aggregated over the states to a physical value of wealth \( p^i w^i \).

In the article, \( \sum_i, \sum_s, \sum_k \) will denote unambiguously the explicit sums \( \sum_{i=0}^{I}, \sum_{s=1}^{S}, \sum_{k=1}^{K} \).

4 Assumptions

The first triplet of hypotheses specifies the financial asset fundamentals (returns).

**Assumption 1** For any country \( i \) and any state \( s \), \( \sum_k R_{sk}^i > 0 \).

**Assumption 2** For any country \( i \) and any financial asset \( k \), \( \sum_s R_{sk}^i > 0 \).

When Assumption 1 fails, there is a country \( i \) and a state \( s \) where any financial asset \( k \) yields \( R_{sk}^i = 0 \). In this case, the representative agent of country \( i \) will enjoy her endowment in the state \( s \).

When Assumption 2 fails, there is an asset \( k \) yielding \( R_{sk}^i = 0 \) in any state of nature \( s \) in country \( i \); the representative agent \( i \) will refuse to buy this financial asset. The following assumption is stronger and implies Assumption 2.
Assumption 3 For any country $i$ and any portfolio $x^i \neq 0$, the portfolio return is nonzero: $R^i x^i \neq 0$.

Assumption 3 means that there are no nonzero portfolios with a null return in any state of nature. In other terms, whatever country $i$ we consider, rank $R^i = K$ and the $K$ financial assets are not redundant.\(^2\)

The second triplet specifies the real fundamentals (endowments and preferences).

Assumption 4 Endowments are positive: $e^i_s > 0$ for any agent $i$ and any state $s$.

Assumption 5 Beliefs are positive: $\pi^i_s > 0$ for any agent $i$ and any state $s$.

This assumption means that any representative agent considers each state possible.

Eventually, preferences are required to satisfy regular assumptions.

Assumption 6 For any agent $i$, the utility function is $u^i (w^i_s) = w^i_s$ for $w^i_s \in \mathbb{R}$.

5 Preferences

The agents’ behavior comes down to a saving diversification. In the state $s$, agents exchange their endowments according to their portfolio:

$$w^i_s = e^i_s + R^i_s x^i$$  \hfill (1)

Preferences of agent $i$ are rationalized by a Von Neumann-Morgenstern utility function weighted by subjective probabilities: $\sum_s \pi^i_s w^i_s$, where $w^i_s$ is her wealth. Thus, $w^i_s$ is permitted to become negative in some states of nature and the utility function is defined on the whole space: $w^i_s \in \mathbb{R}$. The portfolio set $X^i$ coincides with $\mathbb{R}^K$.

In the first period, agent $i$ diversifies her portfolio in order to satisfy her welfare under the financial budget constraint:

$$\max_{x^i \in \mathbb{R}^K} \sum_s \pi^i_s (e^i_s + R^i_s x^i) \quad (2)$$

$$qx^i \leq 0$$

The right-hand side of the budget constraint is zero because we consider the agents’ net purchases.

\(^2\)Market completeness means that the columns of $R^i$ span the whole space $\mathbb{R}^S$ (rank $R^i = S$) and implies that a full insurance is possible. Redundancy of financial assets means that $\dim \ker R^i > 0$, that is $K > \text{rank } R^i$. When capital markets are complete and financial assets are not redundant, we have $K = S = \text{rank } R^i$. In this case, the return matrix is square and invertible.
6 Arbitrage

In economics, arbitrage is the practice of taking advantage of a price difference between two markets. Thereby, there is room for arbitrage when the law of one price is violated.

In the international trade literature, arbitrage is possible when the same good has different prices in different countries, expressed in the same currency. Let $\tau_i^s$ be the exchange rate between country 0 and country $i$ in state $s$. Let $p_i^s$ be the good market price in country $i$ and state $s$. The following condition is the usual no-arbitrage condition for good markets.

**Condition 1** (*Purchasing Power Parity*) $\tau_i^s = p_i^s / p_0^s$ for any $i$ and any $s$.

This condition says that, given the exchange system $(\tau_i^s)$, the goods market prices satisfy the PPP.

In finance, arbitrage is possible when the same asset does not trade at the same price in two markets. The following condition is the usual no-arbitrage condition for financial asset markets.

**Condition 2** (*No Arbitrage NA0*) There exists a vector $q$ such that, for any agent $i$ and any portfolio $x_i \in \mathbb{R}^K$ with $R_i^s x_i \geq 0$ for any $s$ and $R_i^t x_i > 0$ for some $t$, we have $qx_i > 0$.

Such a vector $q$ is called no-arbitrage price system or, more shortly, no-arbitrage price. We have the following result:

**Proposition 3** A price $q$ of a financial asset is a no-arbitrage price if and only if there exists a $S \times (1 + I)$ matrix $(\mu_i^s)$ with $\mu_i^s > 0$ for any $i$ and any $s$, such that, for any $i$, $\mu_i^t R_i^s = \lambda^0 R_0^s = q$ where $\mu_i = (\mu_i^1, \ldots, \mu_i^S)$ denotes the $i$th row of the matrix.

**Proof.** See [2].

We introduce a stronger no-arbitrage condition.

**Condition 4** (*No Arbitrage NA1*) There are positive numbers $\lambda_i$ such that $\lambda_i \pi^i R_i^s = \lambda^0 \pi^0 R_0^s$ for $i = 1, \ldots, I$.

**Remark 5** NA1 implies the existence of a no-arbitrage price $q$. Simply choose $\mu_i^s = \lambda_i \pi_i^s (> 0$ because of Assumption 5) for any $i$ and any $s$, and apply Proposition 3.

7 Risk neutrality, Pareto optimality and exchange rates

**Definition 6** (*net trade*) A matrix of portfolios $x$ is a net trade if $\sum x_i = 0$. 
Definition 7 (Pareto) A portfolio matrix \( x \) is a Pareto allocation if it is a net trade and there exists no other net trade \( y \) which satisfies \( \pi^i R^j y^j \geq \pi^j R^i x^i \) for any \( i \), and \( \pi^j R^i y^i > \pi^i R^j x^j \) for some \( j \).

Definition 8 The pair \((q, x)^*\) is an equilibrium with transfers for the capital market if \( \sum_i x^i = 0 \) and, for any \( i \), \( \pi^i R^j y^j > \pi^j R^i x^i \) implies \( q^* y^i > q^* x^i \).

Proposition 9 In the risk-neutral model, if \( NA_1 \) is satisfied for some positive \( x \), we can define the exchange rates \( \tau \) with prices given by (3) that are \( NA_1 \).

Proof. Let \((q, x)^*\) be an equilibrium with transfers for the capital market. Then, from the Minkowski-Farkas Lemma, we have \( q^* = \lambda^* \pi^i R^j \) with \( \lambda^* > 0 \) for any \( i \). Since \( \sum_k q^*_k = 1 \), it follows

\[
q^*_k = \frac{\sum_i \pi^i R^i_{sk}}{\sum_h \sum_s \pi^i R^i_{sh}}
\]

We will now characterize the optimal allocations for the case of risk neutrality. The following result is well known and we omit the proof.

Proposition 10 If \( x^* \) is a Pareto optimum, then it is an equilibrium with transfers with prices given by (3) that are \( NA_1 \).

Proposition 11 In the risk-neutral model, if \( NA_1 \) is satisfied for some positive vector \( \lambda^* \) then an allocation is Pareto optimal if, and only if, it is a net trade.

Proof. Any Pareto allocation is a net trade, from Definition 7 of Pareto optimality. Let \( x \) be a net trade. Suppose that \( x \) is not a Pareto allocation. In this case, there exists a net trade \( y \) such that \( \pi^i R^j y^j \geq \pi^i R^j x^i \) for any \( i \) and \( \pi^j R^i y^i > \pi^j R^i x^j \) for some \( j \). Thus, \( \sum_i \lambda^* \pi^i R^j y^j > \sum_i \lambda^* \pi^i R^i x^i \), that is \( 0 = \lambda^0 \pi^0 R^i \sum_i y^i > \lambda^0 \pi^0 R^0 \sum_i x^i = 0 \), a contradiction. In the financial literature, some authors endogenize the exchange rates by using Pareto allocations (see Dumas [3] among others). We can do the same in our risk-neutral model. From Proposition 11, any net trade \( x^* \) is Pareto optimal. \( x^* \) solves the program \( \max_{x^i} \pi^i (e^i + R^i x^i) \) under the financial constraint \( q^* x^i \leq q^* x^i \) for any \( i \).

Let us assume that \( w^{i*} = e^i + R^i x^{i*} \) and \( w^i = e^i + R^i x^i \) for any \( i \). The good price system \( p^* \) must satisfy \( p^* w^{i*} = p^* e^i + q^* x^{i*} \). Indeed \( w^{i*} \) solves the program \( \max_w \pi^i w^i \) under the budget constraint \( p^* w^i \leq p^i w^{i*} \) for any \( i \). Then, we have \( p^* = \lambda^i \pi^i \) for any \( i \) and \( q^* = \lambda^i \pi^i R^i = p^i R^i \).

We can define the exchange rates \( \tau^{i*} \) between country \( i \) and country 0 in any state \( s \):

\[
\tau^{i*} = \frac{p^{i*}_s}{p^*_s} = \frac{\lambda^i \pi^i_s}{\lambda^0 \pi^0_s}
\]
Normalizing the price $q^*$ with $\sum_k q^*_k = 1$, we get, for any $i$ and any $s$, 

$$
q^*_s = \frac{\pi^i_s}{\sum_k \sum_t \pi^i_t R^i_{tk}}
$$

Thus, 

$$
\tau^i_s = \frac{\pi^i_s}{\sum_k \sum_t \pi^0_t R^0_{tk}} \sum_k \sum_t \pi^0_t R^0_{tk}
$$

(4)

The exchange rates do not depend on the net trade of financial assets, only on the beliefs and on the returns. Risk neutrality allows us to compute explicitly the optimal exchange rates. When risk neutrality fails (that is risk aversion is different from one), the exchange rates depend on individual allocations through the marginal utilities, and it is no longer possible to provide explicit solutions.

If the beliefs are the same across the countries ($\pi^i = \pi^0$ for any $i$), (4) simplifies to:

$$
\tau^i_s = \frac{\sum_k \sum_t \pi^0_t R^0_{tk}}{\sum_k \sum_t \pi^0_t R^i_{tk}} = \text{total expected return in country 0} / \text{total expected return in country } i
$$

The important question we address is whether the international good trade is balanced, that is 

$$
\sum_i p^i_s w^i_s = \sum_i p^i_s e^i_s 
$$
or, equivalently, 

$$
\sum_i p^i_s R^i_s x^i_s = 0 
$$
for any $s$ with net trade $x^i_s$ and exchange rate system $\tau^i_s$.

But since any net trade is Pareto optimal and hence an equilibrium with transfers, and since the financial asset and commodity prices associated with the Pareto optima are unique (up to a positive scalar), it makes sense to require that the condition $\sum_i p^i_s R^i_s x^i_s = 0$ for any $s$ holds for any net trade $x^*$. We can summarize the results in the following theorem.

**Theorem 12** Assume the NA1 holds for some vector $\lambda^*$. Then an allocation is Pareto optimal if and only if it is a net trade. Any Pareto optimum is an equilibrium with transfers. The financial market prices $q^*_k$ and commodity prices $p^i_s$ are unique (up to a positive scalar):

$$
q^*_k = \frac{\sum_s \pi^i_s R^i_{sk}}{\sum_h \sum_s \pi^0_s R^0_{sh}} \text{ and } p^i_s = \lambda^i \pi^i
$$

Define the exchange rate system by $\tau^i_s = \lambda^i \pi^i / (\lambda^0 \pi^0)$ for any $i$ and any $s$.

1. Then $\tau^i_s = p^i_s / p^0_s$ for any $i$ and any $s$.

If the trade is balanced for any Pareto optimum, then

$$
R^0_s = \tau^i_s R^i_s
$$

(5)

for any $i$ and any $s$.

2. Conversely, assume the financial asset markets are complete and the assets are nonredundant. If $R^0_s = \tau^i_s R^i_s$ for any $i$ and any $s$, then $\tau^i_s = p^i_s / p^0_s$ for any $i$ and any $s$ and the trade is balanced for any Pareto optimum.
Proof. (1) The trade is balanced for the net trade \( x_{0s}^* = -x^i_x \) and \( x^j_x = 0 \) for any \( j \neq 0, i \). The trade balance \( \sum_j p_j^i w_j^i = \sum_i p^i_x e_i^x \) is equivalent to \( \sum_i \tau^i_x R^0_s x^i_x = 0 \) and we find \( (R^0_s - \tau^i_x R^i_s) x^0_{0s} = 0 \) for any \( s \). With \( x_{0s}^* = 1 \) and \( x^h_{0s} = 0 \) for any \( h \neq k \), we get \( R^0_{sk} = \tau^i_x R^i_{sk} \) for any \( k \).

(2) By assumption, \( \lambda^i_x \pi^i_x R^i = \lambda^0_x \pi^0_x R^0 \) for any \( i \) or, equivalently, under (5), \( \lambda^i_x \pi^i_x R^0_s / \tau^i_x = \lambda^0_x \pi^0_x R^0_s \) for any \( i \). Since \( R^0 \) is invertible, we get \( \lambda^i_x \pi^i_x / \tau^i_x = \lambda^0_x \pi^0_x \) or, equivalently, \( \tau^i_x = \pi^0_x / \pi^i_x \). Let \( x^* \) be a Pareto optimum which is a net trade allocation. Then, \( 0 = p^0_x R^0_s \sum_i x^i_x = \sum_i p^0_x \tau^i_x R^i_s x^i_x = \sum_i p^i_x R^i_s x^i_x = \sum_i p^i_x (w^i_x - e^i_x) = 0 \) and the trade is balanced. \( \blacksquare \)

Condition 13 (Uncovered Interest Rate Parity) There exists a matrix of exchange rates \( \tau^i_x \) such that \( R^0_{si} = \tau^i_x R^i_s \) for any \( s \) and any \( k \).

The UIRP says that under the exchange system \( \tau^i_x \), the returns in any country \( i \) equal the returns in country 0. Write the condition \( R^0_{si} = \tau^i_x R^i_s \) as \( p^0_x R^0_s = p^i_x R^i_s \) for any \( i \). If all the countries have a common money, the condition UIRP says that the returns of the countries are the same if they are expressed in this money.

Remark 14 Theorem 12 states the equivalence between PPP and UIRP when NA1 holds, the financial markets are complete and the trade balance is satisfied for any Pareto optimum commodity prices. Indeed, if we define the exchange rate system \( \tau^i_x \) by \( \tau^i_x \equiv \lambda^i_x \pi^i_x / (\lambda^0_x \pi^0_x) \) for any \( i \) and any \( s \), clearly PPP holds with these exchange rates.

Now, if we only require that the international good market clears for some Pareto optima (not for all the Pareto optima), the question is whether these Pareto allocations are no-trade.

In the following example where UIRP does not hold, only the no-trade allocations turn out to be the Pareto allocations that clear the international good market under these exchange rates.

8 Example

We consider a case where NA1 is satisfied on the international capital market, PPP holds but UIRP does not hold. We show that the only individually rational Pareto allocation clearing the international good markets is a no-trade allocation.

Consider an exchange economy with one consumption good, two countries: \( i = 0, 1 \), two assets: \( k = 1, 2 \), and two states of nature: \( s = 1, 2 \). The matrices of returns on assets are:

\[
R^0 = \begin{pmatrix}
R^0_{01} & R^0_{02} \\
R^0_{11} & R^0_{12}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
1 & 3
\end{pmatrix}
\]

\[
R^1 = \begin{pmatrix}
R^1_{01} & R^1_{02} \\
R^1_{11} & R^1_{12}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
3 & 4
\end{pmatrix}
\]
and the beliefs are:

\[ \pi = \begin{bmatrix} \pi_1^0 & \pi_2^0 \\ \pi_1^1 & \pi_2^1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \]

The no-arbitrage condition in the asset markets holds:

\[
\begin{align*}
\lambda^0 \left( \pi_1^0 R_{11}^0 + \pi_2^0 R_{21}^0 \right) &= \lambda^1 \left( \pi_1^1 R_{11}^1 + \pi_2^1 R_{21}^1 \right) \\
\lambda^0 \left( \pi_1^0 R_{12}^0 + \pi_2^0 R_{22}^0 \right) &= \lambda^1 \left( \pi_1^1 R_{12}^1 + \pi_2^1 R_{22}^1 \right)
\end{align*}
\]

with \( \lambda^0 = \lambda^1 \). Let \( \lambda^0 = \lambda^1 = 3 \). Applying the formula \( p_s^* = \lambda^i \pi_i^* \), we find the good prices:

\[ p^* = \begin{bmatrix} p_1^0 & p_2^0 \\ p_1^1 & p_2^1 \end{bmatrix}^* = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{6} \]

and the exchange rates:

\[ (\tau_1^1, \tau_2^1)^* = \left( \frac{\lambda^1 \pi_1^1}{\lambda^0 \pi_1^0}, \frac{\lambda^1 \pi_2^1}{\lambda^0 \pi_2^0} \right) = \left( 2, \frac{1}{2} \right) \]

We have obviously \( p_s^* = \tau_s^* p_s^0 \) for any \( s \) but do not have \( R_s^0 = \tau_s^1 R_s^1 \) for any \( s \). We will see that the no-trade allocation is the unique individually rational Pareto allocation that clears the international good markets under these exchange rates. Indeed, focusing on an individually rational net trade \( x \), the prices \( p^* \) in (6) and

\[ q^* = \lambda^0 \pi^0 R = \lambda^1 \pi^1 R = (3, 6) \]

Now, the trade is balanced if and only if \( \sum_i p_i^* R_i^i x^i = 0 \), \( s = 1, 2 \). Since \( x \) is a net trade, we then have \( x^1 = -x^0 \) and \( (p_1^* R_1^0 - p_1^* R_1^1) x^0 = 0 \) with \( s = 1, 2 \), or, equivalently, \( (R_1^0 - \tau_1^1 R_1^1) x^0 = 0 \). Vectors \( R_1^0 - \tau_1^1 R_1^1 \) are collinear. Since \( x \) is an equilibrium with transfers allocation we must have \( q_1^1 x^0 + q_2^2 x^2 = 0 \). Let us focus on equations \( q^* x^0 = 0 \) and \( (R_1^0 - \tau_1^1 R_1^1) x^0 = 0 \) (state 1) or, more explicitly, on the system \( M x^0 = 0 \) with

\[ M = \begin{bmatrix}
q_1^1 & 0 & q_2^2 \\
R_{11}^0 - \tau_1^1 R_{11} & R_{12}^0 - \tau_1^1 R_{12} & \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix} \]

\[ \det M \neq 0 \] implies \( x^0 = 0 \) and, finally, \( x^i = 0 \) for any \( i \) (no trade).

We have exhibited an example where the set of Pareto optima compatible with the explicit exchange rates is the set of no-trade allocations. In this respect, the assumption of the agents’ risk neutrality in order to endogenize exchange rates through Pareto optima gives a solution, but, unfortunately, it is no-trade and does not mimic what we observe in the international good market.

### 9 Beyond risk neutrality: a remark

Our results hold in a risk neutral world. To conclude, it is worthy to consider the case of risk aversion and bridge our model and Dumas [3], that is a two-country economy with a single financial asset and no exchange of commodities.
We provide a formal proof that, in such a simplified world, the only possible equilibrium (with transfers) is no-trade.

Consider two symmetric countries, 0 and 1, identically weighted in the planner’s program: risk-averse agents have no endowments and share the same preferences (utility functions and beliefs) in both the countries. The respective wealths are simply given by \( c_s^i = R_s^i z^i \) for any \( i \). As in Dumas [3], these countries no longer exchange commodities but directly financial assets and the exchange rate is defined as a ratio of foreign and domestic asset prices. More precisely, we assume that the financial assets are exchanged in terms of currency of country 0. In this case, if country 1 buys a quantity \( z_1 \) of asset, its value in currency of this country is \( \zeta z_1 \), where \( \zeta \) is the exchange rate. The net trade \( x_0 + x_1 = 0 \) in terms of quantities is now replaced by \( z_0 + \zeta z_1 = 0 \) in terms of values.

**Theorem 15** In the case of a single financial asset (\( K = 1 \)) and risk aversion (\( u \) strictly concave), under UIRP, the only equilibrium with transfers (Pareto optimum) is no-trade.

**Proof.** The Pareto problem between the two countries consists in maximizing the objective

\[
V (z^0, z^1) = \sum_s \pi_s u \left( R_s^0 z^0 \right) + \sum_s \pi_s u \left( R_s^1 z^1 \right)
\]

under \( z^0 + \zeta z^1 = 0 \), where \( u \) is assumed to be strictly concave.

The program writes max \( z_1 \) \( V (-\zeta z^1, z^1) \), that is

\[
\max_{ z_1 } \left[ \sum_s \pi_s u \left( -\zeta R_s^0 z^1 \right) + \sum_s \pi_s u \left( R_s^1 z^1 \right) \right]
\]

The first-order conditions become

\[
\zeta = \frac{\partial V / \partial z_1}{\partial V / \partial z_0} = \frac{\sum_s \pi_s R_s^1 u' \left( R_s^1 z^1 \right)}{\sum_s \pi_s R_s^0 u' \left( R_s^0 z^0 \right)}
\]

Under UIRP, we have \( \zeta R_s^0 = R_s^1 \) and, thus, \( z^0 = -\zeta z^1 \) and \( R_s^0 z^0 = -R_s^0 \zeta z^1 = -R_s^1 z_1 \).

Then,

\[
\sum_s \pi_s R_s^1 u' \left( R_s^1 z^1 \right) = \zeta \sum_s \pi_s R_s^0 u' \left( R_s^0 z^0 \right)
\]

\[
\sum_s \pi_s R_s^1 u' \left( R_s^1 z^1 \right) = \sum_s \pi_s \zeta R_s^0 u' \left( -R_s^1 z^1 \right)
\]

\[
\sum_s \pi_s R_s^1 u' \left( R_s^1 z^1 \right) = \sum_s \pi_s R_s^1 u' \left( -R_s^1 z^1 \right)
\]

Since \( u' \) is strictly decreasing, this is possible iff \( R_s^1 z^1 = -R_s^1 z^1 \) for any \( s \), that is, since \( R_s^1 > 0 \) for any \( s \), iff \( z^1 = 0 \) and, eventually \( z^0 = 0 \) (no trade).
Let us found on a theoretical ground the limit case of risk neutrality considered also by Dumas through numerical simulations: \( u(R^t z^t) = R^t z^t \). For simplicity, focus on the riskless case: \( R^s_t = R^t_0 \equiv 1 + r^t \) for any \( s \) and any \( t \). Then

\[
\varepsilon \equiv \zeta - 1 = \frac{\sum_s \pi_s R^1_s}{\sum_s \pi_s R^0_s} - 1 = \frac{1 + r^1}{1 + r^0} - 1 \approx r^1 - r^0
\]

where \( \varepsilon \) represents the deviation from the Law of One Price (LOP).

**10 Conclusion**

Under the assumption of risk neutrality, when the UIRP does not hold, the exchange rates can be explicitly determined and depend only on the agents’ beliefs and the returns on financial assets. Unfortunately, the solution corresponding to these rates may be no-trade. In this respect, our note indicates that the assumption of the agents’ risk neutrality (in order to get an explicit solution) is a problem when we consider an equilibrium on the international good and capital markets. Indeed, if the trade balance is satisfied for any Pareto optimum commodity prices (that is a demanding condition) and other milder assumptions are satisfied, the necessary conditions that clear international good and capital markets are: PPP for the good markets and the UIRP for the international capital market. These conditions are also sufficient when the financial markets are complete.

In a risk-averse economy à la Dumas [3] with a single financial asset and no exchange of commodities, the only possible equilibrium with transfers turns out to be no-trade.

**References**


