Data-Driven Robust Optimization with Application to Portfolio Management

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Abstract:

Portfolio optimization results are strongly dependent on the model parameters. To circumvent this shortcoming, this paper proposes a new modeling approach to address data uncertainty. The model offers full control over the degree of conservatism and underlines its interaction with robustness for a range of extreme market situations. Without prior assumptions on data generating process, we develop a probabilistic guarantee on the optimality of the solution. Unlike previous measures that depend solely on the uncertainty model, our measure is also sensitive to the solution and the investment horizon. We provide an example on international stock indexes investment. Computational experiments and ex-post analysis provide evidence for the potential and the effectiveness of our model.

Keywords: Portfolio protection, Robust optimization, Multivariate tail dependence, Non-parametric predictive inference.
1. Introduction

Since Markowitz’s (1952) mean-variance model, the asset allocation problem has been extensively investigated in finance. At the same time, another strand of research has focused on investment protection using safety-first criteria (Roy, 1952; Telser, 1955 and Katoaka, 1963). Such models seem to describe better investors’ expectations, especially during strong market downturns (Harlow, 1991 and Brogan and Stidham, 2005). Traditionally, portfolio optimization relies on either distributional assumptions or first moments’ estimation of returns. In practice, exact distributions are rarely known. Needless to say, small changes in these input parameters lead to non-efficient portfolios.¹ In this respect, Michaud (1989) and Chopra and Ziemba (1993) document that portfolios obtained from sample mean and covariance matrix estimations show poor out-of-sample performance. These observations motivate the need for models that are immune to data uncertainty.

Robust optimization handles this issue by specifying a deterministic uncertainty set for the parameters based on limited information about their values. It merges the steps of estimating the random parameters and finding a solution that remains feasible for any realization of the uncertain coefficients within prescribed uncertainty sets. There have been numerous attempts to apply robust optimization results to the asset allocation problem. Lobo and Boyd (2000) provide an introduction to the robust portfolio selection using several uncertainty sets of returns. Costa and Paiva (2002), Goldfarb and Iyengar (2003) and Erdogan et al. (2006) study the robust portfolio optimization for the mean-variance model. El Ghaoui et al. (2003) investigate robust portfolio selection using a worst-case Value-at-Risk measure. Similarly, Zhu and Fukushima (2009) consider a worst-case CVaR where only partial information on the probability distribution of returns is given. DeMiguel and Nogales (2009) propose a new approach for portfolio selection by minimizing robust estimators of portfolio risk. There is often a lack of probabilistic justification motivating the choice of uncertainty models. Basic approaches construct intervals around point estimates of uncertain parameters or use past realizations. More generally, uncertain parameters may vary in continuous intervals or in convex sets (Ben-Tal et al, 2000; Goldfarb and Iyengar, 2003; Bertsimas and Sim, 2004; Chen et al, 2007 and Averbakh and Zhao, 2008). Uncertainty may also be expressed by a finite set of scenarios (Mulvey et al, 1995; Kouvelis and Yu, 1997; Bertsimas and Thiele, 2006; Bertsimas and Brown, 2009; Natarajan et al, 2009 and Bertsimas et al., 2010).

A first robust model proposed by Soyster (1973) specifies intervals’ bounds of uncertain parameters. By ignoring information on correlations, this basic model generates an over-conservative solution in that it gives up too much of optimality in order to ensure robustness (Quaranta and Zaffaroni, 2008). To control the price of robustness, recent studies have modeled asymmetries and dependencies among uncertain parameters. For example, Bertsimas and Sim (2004) construct a model that controls the level of conservativeness.¹ Chen et al. (2007) propose a generalized framework, which captures the distributional asymmetry and preserves the convexity and the tractability of the initial optimization problem. In the same way, Bertsimas and Sim (2004) develop an affine model of perturbation factors to describe correlations between the parameters. Miao et al. (2007) construct an autoregressive mobile average model (ARMA) to estimate the coefficients associated with the perturbation factors. A second important issue in robust optimization is to identify probabilistic guarantees against lower guaranteed return violation. Chen et al. (2007), Bertsimas and Sim (2004) and Ben-Tal and Nemirovski (2000)

¹ Other portfolio risk measures such as VaR and CVaR are also affected by this shortcoming (Fabozzi et al., 2010).
develop measures for interval-based model. To the best of our knowledge, this issue has not been addressed for discrete uncertainty model.

In this paper, we propose a scenario-based methodology to address data uncertainty for the portfolio protection problem. This approach offers full control on the degree of conservatism by adjusting lower tail dependencies between returns. Our main goal is to study the interactions between the conservatism and the robustness of optimal portfolios for various uncertainty patterns. As a second objective, we assess the probability guarantee of the robust portfolio using a nonparametric predictive inference technique. Unlike previous measures that depend fully on the uncertainty model, our measure is also sensitive to the optimal solution and the investment horizon. Such property is well suited when new deterministic constraints are added to the initial problem.

The remainder of the paper is organized as follows. Section 2 introduces the portfolio protection problem and discusses its robust counterpart for various uncertainty models. Section 3 presents the new uncertainty model and formulates the problem. Section 4 describes the probabilistic guarantee of the robust portfolio. The results of computational experiments and ex post analysis are discussed in section 5. Section 6 concludes with suggestions for further work.

2. Robust portfolio protection

Let us consider an asset allocation on \( J \) risky assets taking place at \( t = 0 \) and kept unchanged until the end of the investment horizon. The vectors of random return and assets weights are denoted by \( \mathbf{r} = (r_1, \ldots, r_J) \) and \( \mathbf{x} = (x_1, \ldots, x_J) \), respectively. The set \( \{ \mathbf{x} \in [0,1]^J \mid \mathbf{x}^\top \mathbf{e} \leq 1 \} \) refers to no possibility of short sales and budget constraint. Let us denote by \( r_{\min} \) the minimum guaranteed return. The investor's goal is to choose a portfolio with the highest \( r_{\min} \) while limiting at the same time the probability \( \alpha \) of being under this protection level. This setting was formerly exposed by Kataoka (1963). It refers to safety-first criterion, which focuses on downside risk. In practice, investors also attach importance to profitability. Ding and Zhang (2009) integrate a minimum expected return constraint to the Kataoka model whereas Bienstock (2007) introduces a risk aversion parameter that balances the two objectives. When the joint distribution of returns is known with certainty, we express the problem using a chance constraint:

\[
\begin{align*}
\max & \quad \mathbf{r}^\top \mathbf{x} + \lambda r_{\min} \\
\text{s.t} & \quad P(\mathbf{r}^\top \mathbf{x} \leq r_{\min}) \leq \alpha \\
& \quad A \mathbf{x} \leq b
\end{align*}
\]  

where \( \alpha \) is a predetermined critical level to achieve the target, \( \mathbf{r} \) is a statistical estimate of the expected returns and \( \lambda \in \mathbb{R} \) stands for the risk aversion parameter.\(^2\) The deterministic constraints \( A \mathbf{x} \leq b \) may describe, for example, a maximum allocation per asset or per group of assets. Even if we know the distributions, it is still computationally challenging to evaluate the chance constraints. When the joint distribution is not exactly known, stochastic optimization is no longer appropriate. It is unlikely to achieve the desired optimal value because the unknown extent of adverse variation of parameters. Robust optimization provides an alternative formulation to the problem that can be defined as:

\(^2\) When \( \lambda = 0 \), the investor focuses only on the maximization of the expected return. On the other hand, when \( \lambda \) goes to infinity, the investor seeks to maximize the minimum guaranteed return.
\[
\begin{align*}
\max_{x \in X} & \quad F^T x + \min_r \left[ \lambda \left( r^T x \right) \right] \quad \forall \ r \in U \\
\text{s.t} & \quad A x \leq b
\end{align*}
\] (2)

where \( U \) is the uncertainty set of returns.

The optimal solution of program (2) is called a robust solution. Applying this method requires the construction of the robust counterpart of (2). The complexity of such a task depends on the structure of the original program under consideration and the uncertainty model. In this study, we focus essentially on linear programs that reflect the specific structure of the Kataoka model. Next, we analyze the robust counterpart of (2) from some specific uncertainty models.

### 2.1 Hypercube uncertainty model

The hypercube uncertainty set proposed by Soyster (1973) is by far the simplest model to implement. It assumes that individual return vary in bounded interval independently from other returns. The true value \( r_j \) of an uncertain return is then given by:

\[
r_j = \bar{r}_j + z_j \tilde{r}_j \quad \forall \ j = 1, \ldots, n
\] (3)

where \( \bar{r}_j \) is a statistical estimate of the expected value of \( r_j \), \( \tilde{r}_j \) is a statistical estimate of the maximum distance that \( r_j \) is expected to deviate from the point estimate \( \bar{r}_j \) and \( z_j \) is a deviation factor which varies in the interval \([-1, 1]\). Program (2) can be written as:

\[
\begin{align*}
\max_{x \in X} & \quad x^T \bar{r} + \lambda \left[ \min_{x \in X, r \in U} \left( x^T r \right) \right] \\
\text{s.t} & \quad A x \leq b
\end{align*}
\] (4)

The dual formulation of the inner program leads to the robust form of the program (2): ³

\[
\begin{align*}
\max_{x \in X, u, v} & \quad x^T \bar{r} + \lambda \left[ u^T (\bar{r} + \tilde{r}) - v^T (\bar{r} - \tilde{r}) \right] \\
\text{s.t} & \quad A x \leq b \\
& \quad u - v = x \\
& \quad u \geq 0, \ v \geq 0
\end{align*}
\] (5)

by replacing \( u \) with \( x \) and \( l \), we obtain:

\[
\begin{align*}
\max_{x \in X, \delta, \mu} & \quad x^T \bar{r} + \lambda \left[ x^T (\bar{r} + \tilde{r}) + 2v^T \tilde{r} \right] \\
\text{s.t} & \quad A x \leq b \\
& \quad \delta + \mu_j \geq \tilde{r}_j x_j, \forall j = 1, \ldots, J \\
& \quad \delta \geq 0, \ \mu \geq 0
\end{align*}
\] (6)

Although the Soyster's method admits the highest protection, it is also the most conservative in the sense that the robust solution has the worst objective function value. Applying this model is equivalent to choose an optimal portfolio by assuming extreme realizations of returns.

³ In an attempt to simplify the presentation, we limit ourselves to the most important attributes of the optimization programs. Intermediate steps that describe dual transformation can be found in Gabrel and Murat (2010).
2.2 Budget of uncertainty model

Bertsimas and Sim (2004) propose a model that highlights the tradeoff between the robustness and conservatism of optimal solution. By introducing the parameter \( \Gamma \), called the budget of uncertainty, they control the maximum number of uncertain parameters taking their worst-case realization at the same time. Using the deviation factors to describe uncertainty, one has to solve this program:

\[
\max_{\mathbf{x} \in \mathbb{R}_+^n, \mathbf{r} \in \mathbb{R}_+^m} \mathbf{x}^T \mathbf{F} + \lambda \left( \mathbf{x}^T \mathbf{F} + \min_{\sum_j |\Gamma_j - 1| \leq z_j \leq 1} \mathbf{z}^T (\mathbf{r}) \right)
\]

(7)

Since, \( \mathbf{x} \geq 0 \) deviation factors that deteriorate the objective function are such that \( \mathbf{z} \leq 0 \). Therefore, (7) can be rewritten as follows:

\[
\max_{\mathbf{x} \in \mathbb{R}_+^n, \mathbf{r} \in \mathbb{R}_+^m} \mathbf{x}^T \mathbf{F} + \lambda \left( \mathbf{x}^T \mathbf{F} - \max_{\sum_j \Gamma_j \leq 0, 0 \leq z_j \leq 1} \mathbf{z}^T (\mathbf{r}) \right)
\]

(8)

Using the dual transformation of the inner program, we obtain the robust version of (8):

\[
\max_{\mathbf{x} \in \mathbb{R}_+^n, \mathbf{b} \in \mathbb{R}_+^m} \mathbf{x}^T \mathbf{F} + \lambda \left( \mathbf{x}^T \mathbf{F} + \Gamma_0 \delta + \sum_j \mu_j \right)
\]

s.t

\[
\begin{align*}
A \mathbf{x} & \leq \mathbf{b} \\
\delta + \mu & \geq \mathbf{r} \\
\delta & \geq 0, \quad \mu \geq 0
\end{align*}
\]

(9)

A key issue addressed by Bertsimas and Sim (2004) is the measurement of the probability of non violation of the protection level for a given uncertainty budget \( \Gamma \). They demonstrate that for independently and uniformly distributed deviation factors in the interval \([-1,1]\), this probability is at least equal to:

\[
\Phi \left( \frac{\Gamma_0 - 1}{\sqrt{J}} \right)
\]

(10)

where \( \Phi \) is the normal cumulative distribution function and \( J \) is the number of uncertain parameters.

2.3 Polytope uncertainty model

Polytope uncertainty model can be seen as a generalization of the model Bertsimas and Sim (2004) that define a set of affine restrictions between deviation factors. The robust counterpart of (2) has the following form:

\[
\max_{\mathbf{x} \in \mathbb{R}_+^n, \mathbf{r} \in \mathbb{R}_+^m} \mathbf{x}^T \mathbf{F} + \lambda \left( \mathbf{x}^T \mathbf{F} + \min_{\sum_j \Gamma_j \leq 0, 0 \leq z_j \leq 1} \mathbf{z}^T (\mathbf{r}) \right)
\]

(11)

---

\(^4\) Details on the construction of the robust version using the duality can be found in Gregory et al. (2011).
where $F$ is an $(p \times n)$ matrix and $g$ is an $(p \times 1)$ vector. Since $x \geq 0$, deviation factors that deteriorate the objective function are such that $z \leq 0$. Therefore, one can rewrite (11) as:

$$\max_{x \in \mathbb{R}^n, z \geq 0} x^T \bar{r} + \lambda \left( x^T \bar{r} + \min_{g_j \in \mathcal{G}_j} z^T (\hat{r}x) \right)$$

The robust counterpart is equivalent to:

$$\max_{x \in \mathbb{R}^n, z \geq 0} x^T \bar{r} + \lambda \left( x^T \bar{r} + \sum_{j=1}^n \delta g_j + \sum_{j=1}^n \mu_j \right)$$

$$\text{s.t. } Ax \leq b$$

$$F' \delta + \mu \geq \hat{r}x$$

$$\delta \geq 0, \mu \geq 0$$

(12)

To control the conservatism of the robust solution, one may add new linear restrictions that will reduce the size of the uncertainty set and thus restrict the occurrence of worst-case returns. Conversely, reducing the existing restrictions may enlarge the range of extreme values of returns and leads to a conservative solution.

### 2.4 Discrete uncertainty model

Unlike continuous models, discrete uncertainty models assume that the available information is a finite set of scenarios (Kouvelis and Yu, 1997). Formally, let us define a set of scenarios $\Omega = \{s_1, \ldots, s_T\}$ associated with a multivariate realization of returns belonging to $U = \{r^1, \ldots, r^T\}$. In this case, program (2) can be written as:

$$\max_{x \in \mathbb{R}^n, z \geq 0} x^T \bar{r} + \lambda \left[ \min_{s \in \Omega} x^T r^s \right]$$

(13)

Let $U' = \{z^1, \ldots, z^T\}$ the uncertainty set of deviation factors. The formulation of the problem:

$$\max_{x \in \mathbb{R}^n, z \geq 0} x^T \bar{r} + \lambda \left[ x^T \bar{r} + \min_{s \in \Omega} (z^s)^T (\hat{r}x) \right]$$

(14)

The robust counterpart formulation is obtained by adding as new constraints as the number of scenarios. The minimax program is transformed into a one-level linear program of the form:

$$\max_{x \in \mathbb{R}^n, \theta} x^T \bar{r} + \lambda \left( x^T \bar{r} + \theta \right)$$

$$\text{s.t. } \sum_{j=1}^T z_j^t \bar{r}_j x_j \leq -\theta, \quad t = 1, \ldots, T$$

$$Ax \leq b$$

(15)

**Proposition 1** The robust portfolio obtained with a discrete uncertainty model (15) is a solution of the following convex program:

$$\max_{x \in \mathbb{R}^n, z \in \text{conv}(U')} x^T \bar{r} + \lambda \left[ x^T \bar{r} + \min z^T (\hat{r}x) \right]$$

(16)

where $\text{conv}(U')$ is the convex hull of the uncertainty set ($U'$).

Proposition 1 set up the connection between discrete and interval-based uncertainty models.\(^5\) The convex hull is the smallest convex polytope that contains all scenarios and commonly defined as the intersection of affine half-spaces as in the program (11). Accordingly, an alternative way to solve program (14) is to identify the affine relationships between deviation factors using the convex hull construction (Ben Tal et al, 2008).

\(^5\) The proof of proposition 1 can be found in Bertsimas and Gupta (2011).
3. Problem formulation

3.1 Extensions of the discrete uncertainty model

The main results of robust optimization theory discussed so far are related to the connection between discrete and polytope uncertainty models and the effect of the size of uncertainty set on robustness. Clearly, discrete models better describe the joint behavior of returns, even if they are not enough flexible to control the conservatism of robust solutions. To circumvent this shortcoming, we develop a new technique that gradually modifies the lower tail dependence structure between initial scenarios. Similar to Bertsimas and Sim (2004), this technique controls the number of parameters that will take their worst-case realizations at the same time.

In the numerical example presented in Table 1, we illustrate how to control discrete uncertainty sets. The approach can easily be extended to higher dimensional analysis. Without loss of generality, let $A_i$ denotes the set of historical returns observed for three stocks over four successive periods. It is important to underline the intrinsic dependence structure, especially between extreme values. For instance, note that lowest returns occur over separate periods. Hence, to protect his portfolio by using the uncertainty set $A_0$, an investor has to deal with four moderate adverse scenarios.

Table 1 Illustrates the structural break technique used to construct new uncertainty set of returns. $A_0$ is the initial set of scenarios reflecting observed return of three assets over four successive periods. Sets $A_1$ and $A_2$ include scenarios obtained with one and two structural breaks in the initial dataset, respectively. Worst-case realizations are highlighted in grey.
To introduce the effect of structural breaks on tail dependencies, let us assume that returns of any couple of assets observed at time \( t \) may combine with that of a third asset observed at time \( s \), \( t \neq s \). In doing so, the set \( A_1 \) contains all new combinations obtained following this condition that modifies dependencies between scenarios compared to the initial dataset. Two of the three assets maintain the same temporal occurrence so that two stocks simultaneously reaching their worst-case levels. We pursue the example by building more adverse scenarios assuming that each return observed at a time \( t \) may combine with other returns over distinct periods. The set \( A_2 \) displayed in Table 1 contains all new combinations obtained following this condition. To link these results with the model of Bertsimas and Sim (2004), we define the following sets: \( U_0 = A_0, U_1 = A_0 \cup A_1 \) and \( U_2 = A_0 \cup A_1 \cup A_2 \). The set \( U_2 \) indicates the presence of two structural breaks compared to the initial set of scenarios. The set \( U_1 \) contains more adverse scenarios to the investor than does \( U_0 \). With \( J \) uncertain parameters and \( T \) initial scenarios, we may generate new scenarios by fixing the number of \( K \) unchanged parameters and introducing \( J-K \) structural breaks.

This technique introduces, however, additional difficulties regarding the explicit definition of new uncertainty sets. As shown in the example in Table 1, the size of the uncertainty set increases rapidly with the number of structural breaks. It becomes challenging to list explicitly all the scenarios and make the link with the polytope uncertainty model. Obtaining robust a counterpart formulation leads to an increase in computational complexity and suggests the development of an appropriate framework.

### 3.2 Economic interpretation of the problem

Because of the implicit form of uncertainty sets induced by structural breaks, we use a minimax formulation derived from Bienstock (2007) model. Such setting can be viewed as a sequential zero-sum game between two players (leader) and (follower). The investor can be seen as the leader and the nature as the follower. An interesting question addressed concerns the interaction between the two players. Specifically, how a player can limit the loss induced by the adversary choice? In this sense, we assume that the investor has two control mechanisms. First, he fixes the number of structural breaks in the dataset. The second mechanism is to impose an exogenous constraint that explicitly sets the level of protection. We summarize the structure of the game using the following assumptions:

**Assumption 1:** The investor and the nature act on a non-cooperative way to reach their goals.

**Assumption 2:** The investor fixes the uncertainty model to limit the "nature" power.

**Assumption 3:** As a second control mechanism, the investor may impose an explicit constraint to limit the "nature" power.

**Assumption 4:** The level of protection is always obtained from the smallest number of structural breaks.

---

6 Unlike the standard form of the game introduced by Stackelberg, the players have the same objective function which is maximized for the first and minimized for the second. The pessimistic form of the Stakelberg model better describe our problem. The follower (nature) responds to the investor portfolio decision by choosing the worst-case multivariate scenario, while the investor tries to minimize the loss resulting from the nature choice.
The first assumption presents the general structure of the game. Assumptions 2 and 3 describe the control mechanisms used by the investor, which entail a cost in terms of robustness discussed next. Assumption 4 reflects a priority order in the activation of control mechanisms. The investor first sets the number of structural breaks and then varies the level of protection.

3.3 Formulation and problem solution

Our model attempts to characterize a range of optimal levels of protection for various numbers of structural breaks. In a second step, we determine the intermediate values between two consecutive numbers of structural breaks by activation of a specific constraint that will fix the degree of conservatism. Solutions obtained from this problem will be robust to all scenarios belonging to the set of uncertainty. The formulation of the portfolio protection problem is as follows:

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{J} \bar{x}_j + \lambda \left( \sum_{j=1}^{J} x_j \left( \bar{r}_j + \rho_j \sum_{t=1}^{T} y_{t,j} z_{t,j} \right) \right) \\
\text{s.t} \quad & \mathbf{a} \mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} \quad & \sum_{j=1}^{J} x_j \left( \bar{r}_j + \rho_j \sum_{t=1}^{T} y_{t,j} z_{t,j} \right) \\
\text{s.t} \quad & \sum_{j=1}^{J} y_{t,j} = 1 \quad \forall \quad j=1,\ldots,J, \\
& \sum_{j=1}^{J} w_{j} = 1 \\
& \sum_{j=1}^{J} y_{t,j} - K \leq 0 \quad \forall \quad t=1,\ldots,T, \\
& K w_{t} - \sum_{j=1}^{J} y_{t,j} \leq 0 \quad \forall \quad t=1,\ldots,T, \\
& \sum_{j=1}^{J} x_j \left( \bar{r}_j + \rho_j \sum_{t=1}^{T} y_{t,j} z_{t,j} \right) \leq P L_x \\
& y_{t,j} \in \{0,1\} \quad \forall \quad t,j, \quad w_{t} \in \{0,1\} \quad \forall \quad t.
\end{align*}
\]

The problem (18-27) belongs to the class of bi-level mixed integer linear programs. The upper level program (18-20) is continuous with \(J\) variables and \((I+J)\) constraints. The lower level program (21-27) is discrete with \(T(J+1)\) binary decision variables and \((2T+J+2)\) constraints. In this program, constraint (26) corresponds to the second control mechanism that will be used by the investor to limit the "nature" power. Similar to Bienstock (2007), we use a cutting-plane algorithm to solve the bilevel problem. This algorithm refines iteratively the feasible set by means of linear inequalities (cuts). Such procedures are commonly used to find solutions to MILP problems. The principle is that the original problem is relaxed by ignoring the follower’s minimization. During the procedure, each cut should eliminate as much as possible of the unnecessary part of the feasible region and new vertices are generated. The computation of a worst-case return vector and the robust portfolio can be done iteratively with an update of the set containing the optimal deviation factors as follows:

**Step 1** Maximize the minimum of upper level problem with an initial feasible solution \(z_{0}\), to obtain the optimal solution \(x_{*}\).
Step 2 Solve the lower level problem with $x^*_0$ to get the optimal solution $z^*_0$. If $\|z^*_0 - z_0\| \leq \varepsilon$, the solution $(x^*_0, z^*_0)$ is optimal, otherwise, go to Step 3. (Epsilon is a small positive real number).

Step 3 Maximize the minimum of the upper level problem over the updated set $[z^*_0, \ldots, z^*_n]$ to obtain the optimal solution $x^*_0$ and go to Step 2.

It is worth noting that the level of protection depends on the uncertainty model chosen by the investor. In the literature, there is not a specific rule for selecting a priori uncertainty model. It is necessary to find an additional condition which may reflect the degree of conservatism and robustness.

4. Probability guarantee

Probabilistic guarantee for non-violation of the protection level constitutes a cornerstone of the robust optimization method. Chen et al. (2007), Bertsimas and Sim (2004) and Ben-Tal and Nemirovski (2000) develop probabilistic guarantees for interval-based model. To our knowledge, this issue has not been addressed for discrete uncertainty models. Quantification of uncertainty is mostly done by the use of precise probabilities typically satisfying Kolmogorov’s axioms. Whilst this has been very successful in many applications, it has long been recognized to have severe limitations. Using the nonparametric predictive inference (NPI), derived from Coolen (2010), we construct a model to assess portfolios robustness. Nonparametric predictive inference has proved to be efficient for measuring the probability of outcomes that cannot be done using precise probabilities. It relies on the $\mathcal{A}(n)$ assumption developed by Hill (1968) which gives the probability on the realization of a random quantity, conditional to a set of past observed values. The use of $\mathcal{A}(n)$ together with lower and upper probabilities enable inference without prior information on the dataset. However, this method is not sufficient to derive precise probabilities and provides instead a probability interval. Imprecise probability provides important new methods that promise greater flexibility for uncertainty quantification. Its advantages include the possibility to deal with conflicting evidence, to base inferences on weaker assumptions than needed for precise probabilistic methods, and to allow for simpler and more realistic elicitation of subjective information, as imprecise probability does not require experts to represent their judgments through a full probability distribution, which often does not reflect their beliefs appropriately.

To introduce NPI, let us consider a sequence of $(n+m)$ exchangeable Bernoulli trials, each with “success” and “failure” as possible outcomes. Let $(\tilde{s})$ denotes the observed number of successes in the $(n)$ first trials. Let $Y_n$ denotes the random number of successes in trials 1 to $n$. Because of the assumed exchangeability of trials, a sufficient representation of the data is $Y_n = s$.

Let $Y_{n+m}$ denotes the random number of successes in future trials $(n+1)$ to $(n+m)$. Under these assumptions, Coolen (2010) defines the upper and lower conditional probability of the specific event of having at least $(k)$ success within the $(b)$ future trials as:

$$\Pr^u(h, k | T, s) = \Pr^u(Y_{T+b} \geq k | Y_T = s) = \left[\begin{array}{c} T+h \tfrac{T-h}{T} \end{array}\right]^{s+k} \left[\begin{array}{c} T-s+h-k \tfrac{T-s}{T} \end{array}\right]^{s-k} + \sum_{l=1}^{s+k-1} \left[\begin{array}{c} s-l \tfrac{T-s}{T} \end{array}\right]^{s-l} \left[\begin{array}{c} T-s+h-l \tfrac{T-s}{T} \end{array}\right]^{h-l}$$

Bienstock (2007) introduces some refinements for the basic algorithm described above that reduce the number of iterations needed for convergence.
\[
P^x(\{h,k\} | T, s) = P^x(\{Y_{i,s}^x \geq k | Y_i^n = s\}) = 1 - \left( \frac{T + h}{T} \right)^{s+1} \sum_{s-l}^{l} \left( \frac{T - s + h + 1}{T - s} \right)
\]

where \( P^x \) and \( P^u \) are the upper and lower bounds of the probability interval. For the case where the future horizon \((h)\) is equal to one the probability interval:

\[
P^u(1,k|T,s) = P^u(\{Y_{i,s}^x \geq k | Y_i^n = s\}) = \frac{s}{n+1}
\]

As discussed earlier, the robust solution obtained for a given set of uncertainty will not allow violation of protection level from realized scenarios belonging to this set. Following this condition, we are interested in the particular event of non-violation of the protection level by a future scenario. In the absence of structural breaks in the dataset, applying \((NPI)\) framework to our problem is straightforward. More specifically, knowing that none of the \((T)\) observed scenarios violate the protection threshold, we aim to determine the probability interval for a future scenario to fulfill this condition. To link with the previous developments, we assume that a success refers to a non-violation of the level of protection. To determine the probability interval, one has to apply the formulas (28) and (29) by assuming \((T)\) as the number of scenarios, \((h)\) the number of non-violation in the initial set, \((h)\) is the number of future periods and \((k)\) is the minimum number of non-violation.

Applying the NPI technique in the presence of structural breaks requires to find the number of new generated scenarios that do not violate the protection threshold. To this end, let \(T_1\) denotes the number of scenarios obtained with one structural break in the dataset. Based on assumptions (1-4), we assume that \(T_1\) satisfies the following relationship:

\[
T_1 = T_0 + N_{0,1}
\]

where \(N_{0,1}\) is the number of all intermediate scenarios between the levels of protection obtained respectively from the initial set of scenarios and in the presence of one structural break in the dataset. For \((M)\) structural breaks, the recursion formula gives the following result:

\[
T_M = T_{M-1} + N_{M-1,M} \quad \forall M = 1, \cdots J - K
\]

The listing of intermediate generated scenarios between two consecutive structural breaks is obtained through the activation of constraint (26). More precisely, we apply the pseudo-code presented in Appendix (1). The intermediate number of scenarios is expected to be sensitive to the dependencies between scenarios, the size of the original dataset, the number of uncertain parameters and the number of structural breaks assumed. When the size of the data set increases probability guarantee is improved. A similar result is expected for the dependence structure and the number of uncertainty parameters. Having large historical information improves the lower NPI bound if new dependencies do not deviate significantly from those on the original dataset. Unlike Bertsimas et al. (2004), our measure of probability is solution-dependent. This point is fundamental especially when new deterministic constraints are added to the initial portfolio problem.
5. Computational experiments

This section presents the numerical experimentation of the robust portfolio optimization. First, we describe briefly the data used as input in the uncertainty model. Then, we measure the lower tail dependence between returns for different quantiles.

5.1 Data

Data used in this study are collected from DataStream database. The final sample contains weekly return of nine international stock market indices (as presented in Table 2). The sample period ranges from 01/23/1998 to 12/16/2010 for a total of 669 observations. The first 554 observations (01/23/1998 to 08/29/2008) refer to the in-sample period and are used as input for the uncertainty set. The last 115 observations (09/05/2008 to 12/16/2010) refer to the out-of-sample period and are used for the ex-post effectiveness analysis.

5.2 Summary statistics

Table 2 presents summary statistics of indices returns over the in-sample and out-of-sample periods. Skewness values show that the distributions exhibit negative skew which is indicative of a high frequency of negative returns. On the other hand, positive excess of kurtosis illustrate the greater likelihood of extreme values. From the Kolmogorov-Smirnov test results, assets returns seem to deviate largely from the normal distribution. There are some differences however that should be noted between in-sample and out-of-sample periods. Equity markets are more volatile and their distributions are more skewed to the left over the second period, which covers the global financial crisis of 2008.

Table 2 Summary statistics of returns

Panel A: In-sample

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>JBS</th>
<th>Min</th>
<th>Max</th>
<th>Std Dev</th>
<th>Skew</th>
<th>E-Kurt</th>
<th>K-S</th>
<th>p-val</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>4.14</td>
<td>4.29</td>
<td>-11.60</td>
<td>7.78</td>
<td>16.70</td>
<td>-0.35</td>
<td>5.17</td>
<td>119.95</td>
<td>0</td>
</tr>
<tr>
<td>CAC40</td>
<td>5.99</td>
<td>6.04</td>
<td>-11.42</td>
<td>11.67</td>
<td>20.26</td>
<td>-0.02</td>
<td>4.21</td>
<td>33.76</td>
<td>0</td>
</tr>
<tr>
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<td>6.87</td>
<td>6.87</td>
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<td>13.75</td>
<td>23.23</td>
<td>-0.08</td>
<td>4.73</td>
<td>70.08</td>
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</tr>
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<td>10.59</td>
<td>15.74</td>
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</tr>
<tr>
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<td>6.77</td>
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<td>0.06</td>
<td>5.70</td>
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<tr>
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<tr>
<td>SMI</td>
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<td>3.36</td>
<td>-13.62</td>
<td>17.69</td>
<td>18.94</td>
<td>0.10</td>
<td>9.18</td>
<td>883.14</td>
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<td>IBEX35</td>
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Panel B: Out of sample

<table>
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<th>Mean</th>
<th>JBS</th>
<th>Min</th>
<th>Max</th>
<th>Std Dev</th>
<th>Skew</th>
<th>E-Kurt</th>
<th>K-S</th>
<th>p-val</th>
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</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>1.24</td>
<td>1.35</td>
<td>-18.20</td>
<td>12.03</td>
<td>28.92</td>
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<td>7.98</td>
<td>7.72</td>
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<td>FTSE100</td>
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<td>9.22</td>
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<tr>
<td>OMXS</td>
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<td>-1.06</td>
<td>7.91</td>
<td>136.78</td>
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<td>OMXH</td>
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<td>-1.85</td>
<td>-16.45</td>
<td>10.94</td>
<td>30.67</td>
<td>-0.75</td>
<td>4.62</td>
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</tr>
<tr>
<td>OAS</td>
<td>3.04</td>
<td>3.05</td>
<td>-20.88</td>
<td>14.66</td>
<td>36.14</td>
<td>-0.75</td>
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<td>SMI</td>
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<td>-22.28</td>
<td>14.07</td>
<td>28.68</td>
<td>-0.97</td>
<td>12.36</td>
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</tr>
<tr>
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<td>-0.14</td>
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<td>-21.20</td>
<td>11.74</td>
<td>33.68</td>
<td>-1.05</td>
<td>6.40</td>
<td>76.83</td>
<td>0</td>
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</table>

Table 2 presents summary statistics of the nine indices used in the dataset over the in-sample and the out-of-sample periods. Expected returns are estimated using the arithmetic mean and the Jorion, Bayes and Stein (JBS) estimator, respectively (see Appendix 1). The values in the first five columns are given in percentages. The means and standard deviation are annualized.
There are two parameters that reach their lowest values at the same time during the in-sample period. Therefore, the maximum number of structural breaks that will be assumed for our example is fixed to seven. Table 3 presents the matrix of nonparametric left tail dependence among major stock indices at 1% and 5% quantiles. Tail dependence refers to the co-movement among extreme events, which is not necessarily similar to that among ordinary observations. The results on pairwise tail dependence suggest that extreme returns are positively dependent at the 1% quantile. The FTSE 100 has the highest lower tail dependence with the SMI suggesting that the U.K. market suffers the least co-crashes with the Swiss stock market. To check the robustness of these results and that they do not correspond to outliers, we perform the same analysis at the 5% quantile. The values of dependencies increase significantly and confirm the trend highlighted at the 1% quantile.

Table 3. Lower tail dependence between returns over the in-sample period

Panel A: (In sample) Quantile(1%)

<table>
<thead>
<tr>
<th></th>
<th>CAC40</th>
<th>DAX30</th>
<th>FTSE100</th>
<th>OMXS</th>
<th>OMXH</th>
<th>OAS</th>
<th>SMI</th>
<th>IBEX35</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.333</td>
<td>0</td>
<td>0.5</td>
<td>0.333</td>
<td>0.167</td>
<td>0.167</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>CAC40</td>
<td>0.333</td>
<td>0.556</td>
<td>0</td>
<td>0</td>
<td>0.111</td>
<td>0.444</td>
<td>0.222</td>
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</tr>
<tr>
<td>DAX30</td>
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<td>0.053</td>
<td>0</td>
<td>0.053</td>
<td>0.105</td>
<td>0.211</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FTSE100</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
<td>0.8</td>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OMXS</td>
<td>0.077</td>
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<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OMXH</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OAS</td>
<td></td>
<td></td>
<td>0.071</td>
<td>0</td>
<td>0.429</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SMI</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Panel B: (In sample) Quantile(5%)

<table>
<thead>
<tr>
<th></th>
<th>CAC40</th>
<th>DAX30</th>
<th>FTSE100</th>
<th>OMXS</th>
<th>OMXH</th>
<th>OAS</th>
<th>SMI</th>
<th>IBEX35</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.571</td>
<td>0.464</td>
<td>0.536</td>
<td>0.5</td>
<td>0.357</td>
<td>0.214</td>
<td>0.5</td>
<td>0.321</td>
</tr>
<tr>
<td>CAC40</td>
<td>0.419</td>
<td>0.419</td>
<td>0.302</td>
<td>0.186</td>
<td>0.256</td>
<td>0.488</td>
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<tr>
<td>DAX30</td>
<td>0.232</td>
<td>0.232</td>
<td>0.089</td>
<td>0.143</td>
<td>0.304</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>FTSE100</td>
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<td>0.292</td>
<td>0.375</td>
<td>0.75</td>
<td>0.542</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OMXS</td>
<td>0.18</td>
<td>0.22</td>
<td>0.28</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>OMXH</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>OAS</td>
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<td></td>
<td>0.308</td>
<td>0.231</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>SMI</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.429</td>
</tr>
</tbody>
</table>

Table 3 presents the nonparametric lower tail dependence matrix of returns over the in-sample period at the quantile of 1% and 5%. Dependence coefficients are calculated using the formula defined in appendix 2.
6. Results

6.1 Sensitivity analysis

This section reports and discusses results obtained from numerical experimentation. Figures (1-3) illustrate for various levels of risk aversion the effect of the uncertainty model and the diversification constraints on the protection level. Regardless of the investor’s risk attitude, worst-case shortfall decreases as the number of structural breaks in the dataset increases. With an infinite lambda and no diversification constraints, the protection level is (-6.62%). When the number of structural breaks is the highest, the minimum guaranteed return is (-8.48%). All other things being equal, when the parameter $\lambda$ increases, the protection level decreases. Note that for large $\lambda$, the lowest worst-case shortfall is obtained with a portfolio without diversification constraints. In contrast, for small $\lambda$, we have the opposite result. Investors with low risk aversion are more willing to choose high-return assets, which generally are more volatile and with more extreme negative returns.

Figure 1 Protection level sensitivity to breaks and diversification for $\lambda=10^1$

Figure 2 Protection level sensitivity to breaks and diversification for $\lambda=1$

Figure 3 Protection level sensitivity to associated to breaks and diversification for $\lambda=0$

Figure 4 Induced Budget of uncertainty associated to breaks and diversification for $\lambda=10^8$
To make connection with Bertsimas and Sim (2004) model, Figure 4 depicts the evolution of the budget of uncertainty associated to the worst-case shortfall for $\lambda = 10^3$. In the absence of breaks in the dataset, the value of this function is 5.85. Not all the curves in figure 4 are monotonic decreasingly, as they should be. This pattern is reduced as the diversification and/or the number of structural breaks increases. This result shows the convergence of our model with Bertsimas and Sim (2004). Note that for the same level of protection, there are several robust portfolios obtained for different levels of diversification. Two criteria will be used to evaluate the performance of these portfolios. The first one is the probability of non-violation of the protection threshold. The second criterion is the cost of robustness or the sacrifice of expected return against robustness. These two criteria are examined for our model and that of Bertsimas and Sim (2004), respectively.

### 6.2 Ex-post effectiveness

Figures 5-8 present the probability guarantees and frequencies of non-violation of protection level for two future investment horizons (25 and 50 weeks). We test the performance of robustness measures including and excluding to the investment horizons the period of the financial crisis of 2008. We compare results obtained from NPI and from the probability measure of Bertsimas and Sim (2004). First, we note that the frequency of non-violation for both models is high and it is close to 90%. This result tends to improve as the level of protection becomes more conservative or the investment horizon increases. The number of violations increases significantly when the financial crisis phase is included in the test periods. Overall, there is little difference between the results of the two models compared to the frequency of violations. In terms probability guarantee, we focus mainly on the lower bound. Note that an increase in the absolute level of protection results improves the probability guarantees. The lower bounds of the two probability models systematically frame and all investment horizons frequency of non-infringement. However, that obtained from the non-parametric technique (NPI) seems more accurate. Unlike Bertsimas and Sim, our measure decreases more gradually following the protection and it is sensitive to the investment horizon.

**Figure 5** Probability lower bound and frequency of non-violation obtained from the new model over an investment period of 25 weeks

---

8 For the Bertsimas and Sim (2004) model, the upper limit of the probability of non-violation of the protection level is one.
**Figure 6** Probability lower bound and frequency of non-violation obtained from the new model over an investment period of 50 weeks

**Figures 5 and 6** present the probability intervals obtained from the method (NPI) and the frequency of non-violation of protection level. These measurements cover respectively the investment horizons of 25 and 50 weeks.

**Figure 7** Probability lower bound and frequency of non-violation obtained from Bertsimas and Sim model over an investment period of 25 weeks

**Figure 8** Probability lower bound and frequency of non-violation obtained from Bertsimas and Sim model over an investment period of 50 weeks

**Figures 7 and 8** present the probability lower bound and the frequency of non-violation of the protection level. These measurements are obtained from the model of Bertsimas and Sim (2004) and respectively cover investment horizons of 25 and 50.
6.3 Cost of robustness

We compare the realized returns related to four investment models for three horizons periods of (25, 50 and 75 weeks), which include and then exclude the crisis period. The first model describes a situation of perfect certainty where one seeks a portfolio that maximizes return having all observations. The second model considers the case of an investor that is only interested by the maximization of performance using expected return estimations. The last two models integrate the protection of the portfolio with a lambda of 10. They correspond to the model of Bertsimas and Sim and the one we propose. Results are obtained with a threshold diversification set to three and a number of structural breaks of seven. Overall, there is no significant difference between the last two models. In times of crisis, they give similar results to the certainty case. The investor who focuses on maximizing reward function (lambda = 0) has the lowest realized return. For the non-crisis period, the two models of uncertainty are outclassed by the other two models. All these results show that the loss in terms of profitability remains relatively low. We can conclude on the potential of the discrete model of uncertainty to determine the robust solution and to provide a good estimate of probability of violation of the protection threshold.

Figure 9 Realized returns over different investment horizons (including the crisis period)

![Figure 9](image1)

Figure 10 Realized returns over different investment horizons (excluding the crisis period)

![Figure 10](image2)

Figures 9 and 10 exhibit realized return obtained from four models over the investment horizons of (25, 50 and 75 weeks) including and excluding the crisis period (September - December 2008.). The first model describes a situation of perfect certainty where the investor maximizes its portfolio using realized returns. The remaining models use sample mean of returns estimated from the in-sample period. The second model assumes that investors are only interested in maximizing returns. The last two models are based on a lambda equal 10 and correspond to the model of Bertsimas and Sim and our model.
7. Conclusion

For an investor optimizing a portfolio using a safety-first criterion, this paper provides a robust control approach to mitigate the impact of parameter uncertainty. This issue is of great relevance to portfolio managers since uncertainty leads to unstable portfolio weights and low risk-adjusted returns. Unlike many other robustification approaches, our model makes no assumptions about the distributions of the unknown parameters. As a second advantage, it captures multivariate tail dependence between returns by introducing progressive structural breaks in the dataset and hence controls the conservatism of the solution. Furthermore, our approach gives probability bounds of robustness that depend on both intrinsic characteristics of the uncertainty, optimal solution and investment horizon.

Empirical results highlight the importance of taking into account the uncertain model in the optimization process. The concept of structural breaks fairly describes the behavior of financial assets and reflects the shift of correlations that may occur during bear markets. Single extreme asset movement is often less adverse to investors seeking to guarantee the minimum return than a negative variation of all securities. Obviously, in the second case diversification is no longer effective. The nonparametric probability measure $NPI$ provides accurate prediction intervals and illustrates high ex post performance of robust portfolios. The analysis of trade-off between conservatism and robustness of the solution is also explored. It shows that robust decisions have relatively low cost on the objective function.

The evidence provided in the paper, based on a real data application, suggests that scenario-based models work well in practice and provide a viable and a simple alternative to interval-based ones. Our model could readily be applied to other applications related for instance to the solvency issue, which should be fulfilled regardless risks occurrence. Accordingly, it would be possible to determine capital requirement following a worst-case framework. This setting is expected to reduce the risk of model induced by a flawed choice of probability distributions. We are aware, however, of some weaknesses of our approach. Clearly, much work has to be done to effectively address the problems of outliers. It would be interesting in a future study to examine a robust multistage setting for the portfolio protection problem.

Appendix 1: Algorithm of intermediate scenarios enumeration

Set $(b)$ the number of structural breaks
Find the robust solution $(\hat{x}_b^*)$ for a level of breaks $(b)$ and determine the protection level $PL_b(\hat{x}_b^*)$ (by solving the program (18-27))
Find at the node $(b-1)$ the worst-case shortfall $PL_{b-1}(\hat{x}_b^*)$ for $(\hat{x}_b^*)$ (by solving the program (21-27))
Set $Total\_number = T$
Set $i = 0$

While $b > 0$

$PL_b(\hat{x}_b^*)[i] = PL_b(\hat{x}_b^*)$
solve (21-27) for $(b)$ and with the additional constraint:

$-PL_b(\hat{x}_b^*[i]) \leq -PL_b(\hat{x}_b^*) - \epsilon$ (where $\epsilon$ is a small positive real number).
If $PL_b(x^*_p)[i+1] < PL_{b-1}(x^*_p)$
Set $i = i + 1$
Else
Set $Nb = i$
Set $i = 0$
Set $b = b - 1$
Find at the node $(b-1)$ the protection level $PL_{b-1}(x^*_p) = \left( x^*_p \right)^T r^*_{b-1}$ for $x^*_p$ (by solving the program (21-27))
End
Set Total _nomber = Total _nomber + Nb 
End

Appendix 2: Jorion-Bayes-Stein estimator

The Jorion-Bayes-Stein estimator was proposed by Stein (1956) and further elaborated by Jorion (1986). It relies on the shrinkage technique and the Bayesian framework. It is defined as:

$$r^{oa}_j = \omega \bar{r}_j + (1 - \omega) \bar{r}_j$$

where $r^{oa}_j$ the adjusted asset is mean, $\bar{r}_j$ is the original asset sample mean, $\bar{r}_j$ is the global mean (approximated by the MSCI global index) and $\omega$ is the shrinkage factor. Jorion (1986) estimates the shrinkage factor as:

$$\omega = \frac{(J + 2) \bar{r}}{(J + 2) + T(\bar{r} - r^*_g) \hat{\Sigma}^{-1}(\bar{r} - r^*_g)}$$

where $T$ is the sample size, $J$ is the number of assets, $\hat{\Sigma}$ is the sample covariance matrix, $e$ is a unit vector and $r^*_g$ is a vector of the sample means of returns.

Appendix 3: Lower tail dependence index

Let us consider a pair of random variables $(x, y)$ whose realizations are observed over $T$ periods, $x_t$ and $y_t$ for $t = 1, \ldots, T$. The nonparametric dependency estimator $\tau_{y|x}$ between the extreme values is calculated as the ratio between the number of observations where $x$ and $y$ are jointly extreme and those where only the variable $x$ is extreme. More precisely, one has to set a positive small integer ($k$) and find the $k$ lowest values for the two variables satisfying the following formula:

$$\hat{\tau}_{y|x} = \frac{\sum_{i=1}^T I_{y,x} \text{ and } I_{x,y}}{\sum_{i=1}^T I_{x,y}}$$

where $I_{y,x}$ et $I_{x,y}$ are indicator variables taking 1 if the observation at time $t$ for the variables $x$ and $y$ is strictly inferior to the quantile at the level $(k/T)$.

References:


Krokhmal, P., Palmquist, J. and Uryasev, S. (2002). “Portfolio Optimization with Conditional Value-at-