On the existence of equilibrium in an incomplete financial economy with numeraire assets

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Abstract

This note provides an intuitive and simple proof of the existence of equilibrium in an incomplete financial economy with numeraire assets, when the preferences are represented by concave, strictly increasing functions.

Keywords: General Equilibrium, Incomplete Financial Markets, Numeraire assets

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1 Model

We consider a two period model with a certain first period and an uncertain second period. In the second period, there are finite number of possible states of the world. For convenience all variables for the first period will be denoted with a subscript 0 and the the states of the world in the second period will be denoted by a subscript \( s \in S = \{1, ..., S\} \).

On such a stochastic structure, we consider an economy with a positive finite number of consumers, \( i \in I = \{1, ..., I\} \). There exist \( L \) physical goods, so that a consumption bundle and the commodity set for the \( i \)-th consumer can be defined as
\[
(x^i_0, x^i_1, ..., x^i_S) \in \mathcal{X}^i \subset \mathbb{R}^{L \times (S+1)}
\]
where
\[
\forall s, \forall i, \ x^i_s \in \mathbb{R}^L, \ \mathcal{X} = \prod_{i=1}^{I} \mathcal{X}^i.
\]

The initial endowment of consumer \( i \) is given by \( (\omega^i_0, \omega^i_1, \omega^i_2, ..., \omega^i_S) \in \mathbb{R}^{L \times (S+1)} \), where for each \( i \) and \( s \), we have \( \omega^i_s \in \mathbb{R}^L \). As the consumer \( i \) does not know which state of the nature will occur at the second period, \( (\omega^i_s)_{s=1}^{S} \) can be thought of as a random variable.

Each consumer has a complete preference ordering on the respective commodity set \( \mathcal{X}^i \) so that \( u^i : \mathcal{X} \rightarrow \mathbb{R} \) denotes the utility representation of the preference ordering. Accordingly, \( P^i : \mathcal{X} \rightarrow \mathcal{X}^i \) denotes the preference correspondence that assigns every allocation to the set of consumption bundles which are strictly preferred.

The price system at the spot market is denoted by \( (p_0, p_1, p_2, ..., p_S) \in \mathbb{R}_+^{L \times (S+1)} \), where \( p_s \in \mathbb{R}^L \) for each \( s \). Aside from the spot markets, there also exist financial contracts which allow consumers to allocate their wealth across different states of the world at the second period. There are \( J \) number of such financial contracts indexed by \( j \in J = \{1, 2, 3, ..., J\} \).

The \( i \)-th consumer’s financial portfolio alternatives can be defined by \( (z^i_1, z^i_2, z^i_3, ..., z^i_J) \in \mathcal{Z} \subset \mathbb{R}^J \), where \( \mathcal{Z} = \prod_{i=1}^{J} \mathcal{Z}^i \) and \( z^i_j \) denotes the quantity of asset \( j \) bought in the first period by the consumer \( i \). Clearly, the negative values refer then to the quantities sold.

We denote the prices of the financial contracts by \( (g_1, g_2, g_3, ..., g_J) \in \mathbb{R}_+^J \). For a given price system of commodities \( (p_0, p_1, ..., p_S) \in \mathbb{R}^{L \times (S+1)} \), the returns to the financial contracts are denoted by a vector map \( \mathbf{p} \rightarrow V(\mathbf{p}) \)
where

$$p \rightarrow V(p) = \begin{bmatrix}
    v_{11}(p_1) & v_{12}(p_1) & v_{13}(p_1) & \cdots & v_{1J}(p_1) \\
    v_{21}(p_2) & v_{22}(p_2) & v_{23}(p_2) & \cdots & v_{2J}(p_2) \\
    v_{31}(p_3) & v_{32}(p_3) & v_{33}(p_3) & \cdots & v_{3J}(p_3) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    v_{S1}(p_S) & v_{S2}(p_S) & v_{S3}(p_S) & \cdots & v_{SJ}(p_S)
\end{bmatrix}_{S \times J}$$

with \(J \leq S\).

A financial economy \(\varepsilon\) is then summarized with the tuple \(((X^i, u^i, \omega^i, Z^i)_{i=1}^I, V(p))\) from which the corresponding budget set can be characterized by

$$B(p, q) = \{x^i \in X^i, z^i \in Z^i \mid \forall i, \ p_0(x^i_0 - \omega^i_0) \leq -qz^i, \forall s \in S, \ p_s(x^i_s - \omega^i_s) \leq V_s z^i}\}.$$

To analyze the existence of equilibrium in such an incomplete financial economy with **numeraire assets**, we define a numeraire \(e\) with an \(S \times J\) return matrix \(R\) so that \(V(p)\) can now be defined as follows:

$$V(p) = \begin{pmatrix}
    p_1.e & 0 & 0 & \cdots & 0 \\
    0 & p_2.e & 0 & \cdots & 0 \\
    0 & 0 & p_3.e & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & p_S.e
\end{pmatrix} \cdot R.$$

**Definition 1** A financial equilibrium in such an incomplete financial economy is a pair of actions and admissible prices \(((x^i, z^i)_{i=1}^I, (p^*, q^*))\) that satisfy the following conditions:

(i) for each \(i\), \(x^i \in \arg\max(u^i(x))\) where;

\((x^i, z^i) \in B(p^*, q^*) = \{x^i \in X^i, z^i \in Z^i \mid \forall i, \forall s, \ p_0^0(x^i_0 - \omega^i_0) \leq -q^*z^i, \forall s \in S, \ p_s^0(x^i_s - \omega^i_s) \leq V_s z^i\},\)

or equivalently, in terms of the preference correspondences, where

\(P^i(x^i) \cap B(p^*, q^*) \neq \emptyset,\)
(ii) goods market clears:
\[
\sum_{i=1}^{I} x^{*i} = \sum_{i=1}^{I} \omega^i,
\]
and finally,
(iii) the asset market clears:
\[
\sum_{i=1}^{I} z^{*i} = 0.
\]

The following proposition from Florenzano, et al. (1999) proves the existence of equilibrium in an incomplete financial economy with bounded portfolios. This will prove to be useful in presenting our main result, namely the simple and intuitive proof of the equilibrium with numeraire assets.

**Proposition 1** (Florenzano, et al., 1999) For a given economy \( ((X^i, u^i, \omega^i, Z^i)_{i=1}^{I}, V(p)) \), assume that

(i) \( \forall i, X^i \) is a closed, convex and bounded from below subset of \( \mathbb{R}^{L(i+1)} \),

(ii) \( \forall i, u^i \) is concave and strictly increasing,

(iii) \( \forall i, \omega^i \in \text{int}X^i \),

(iv) \( p \rightarrow V(p) \) is continuous,

(v) \( \forall i, Z^i \) is a closed, convex and bounded from below subset of \( \mathbb{R}^{J} \) and \( 0 \in \text{int}Z^i \).

Then there exists an equilibrium \( (x^i, z^i, p, q) \) such that \( ||p_s|| = 1 \), for every \( s \geq 1 \), where \( ||p_s|| = |p_s(1)| + ... + |p_s(L)| \).

**Proof.** See Florenzano et al. (1999).

2 Main Result: Equilibrium with Numeraire Assets

We prove now the existence of equilibrium in an incomplete financial economy with numeraire assets.

**Theorem 1** For a given financial economy \( \varepsilon \) with numeraire assets, \( ((X^i, u^i, \omega^i, Z^i)_{i=1}^{I}, \mathcal{R}) \), assume that
\[(A1) \forall i, X^i = \mathbb{R}^{L(S+1)},\]

\[(A2) \forall i, u^i is concave and strictly increasing,\]

\[(A3) \forall i, Z^i = \mathbb{R}^J,\]

\[(A4) R is a S \times J matrix and \text{rank} R = J.\]

Then there exists an equilibrium \((x^i, z^i, \bar{p}, \bar{q})\) such that \(||p_s|| = 1\), for every \(s \geq 1\), where \(||p_s|| = |p_s(1)| + \ldots + |p_s(L)|\).

**Proof.** Take \(e = (e_1, e_2, \ldots, e_L) > 0\). For \(||p_s|| \leq 1\), let \(\pi_s = 1 - |p_s|_1 - |p_s|_2 - \ldots - |p_s|_L\), and consider

\[
V^\lambda(p) = \begin{pmatrix}
|p_1|.e + \lambda\pi_1 & 0 & 0 & \ldots & 0 \\
0 & |p_2|.e + \lambda\pi_2 & 0 & \ldots & 0 \\
0 & 0 & |p_3|.e + \lambda\pi_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & |p_s|.e + \lambda\pi_s \\
\end{pmatrix} R.
\]

For any \(\lambda > 0\), \(\text{rank} V^\lambda(p) = J, \forall p\).

Let \(A = \{(x^i) \in \mathbb{R}_{+}^{L(S+1)J} | \sum_{i=1}^{I} x^i_s \leq \sum_{i=1}^{I} \omega^i_s, \forall s \geq 0\}\) and \(A^i\) be the projection of \(A\) on the \(i\)-th component so that

\[
A^i = \{x \in \mathbb{R}_{+}^{L(S+1)} | \exists (x^j)_{j \neq i}, \sum_{j \neq i} x^j_s + x_s \leq \sum_{i=1}^{I} \omega^i_s, \forall s \geq 0\}.
\]

By definition, \(A\) and \(A^i\) are compact and convex. Let \(B\) be the ball of \(\mathbb{R}_{+}^{L(S+1)}\) that contains \(A^i\) in its interior.

**Lemma 1** For any \(i\), there exists a bounded set \(\tilde{Z}^i\) which contains in its interior the set \(\tilde{Z}^i\) that satisfies

\[
p_s(x^i_s - \omega^i_s) = V^\lambda_s z^i, \forall s \geq 1, \text{ with } ||p_s|| \leq 1, (x^i) \in A^i, \forall s, \forall i.
\]

**Proof.** The set \(\tilde{Z}^i\) of each \(z^i\) is uniformly bounded. Suppose that there exists a sequence \((p^n, z^{(n,i)}, x^{(n,i)})\) with \(||z^{(n,i)}|| \to \infty, ||p^n|| \leq 1, \forall s, x^{(n,i)} \in A^i, \forall n\). We can assume that \(\frac{x^{(n,i)}}{||x^{(n,i)}||} \to \tilde{z}^i\) with \(||\tilde{z}^i|| = 1\).

However, note that \(V^\lambda_s \tilde{z}^i = 0, \forall s\), since \(\text{rank} V^\lambda = J\). We have \(\tilde{z}^i = 0\): contradiction. We can choose \(\tilde{Z}^i\) such that \(\tilde{Z}^i\) is in the interior of \(\tilde{Z}^i\).
Now let $\tilde{X}^i = B$, $\forall i$. By Proposition 1, the financial economy $\varepsilon(\lambda) = (X^i, u^i, Z^i, \omega_i)_{i=1}^I, V^\lambda$ has an equilibrium $(x^*i(\lambda), z^*i(\lambda), p^*\lambda, q^*\lambda)$ where $||p^*\lambda|| = 1, \forall s$. Since $u^i$ is strictly increasing, we have $p^*\lambda > 0, \forall s$.

Hence $p^*\lambda \cdot e > 0, \forall s$. For some $\lambda > 0$, consider that

$$V^*\lambda = \begin{pmatrix}
p_1^*\lambda \cdot e & 0 & 0 & \ldots & 0 \\
0 & p_2^*\lambda \cdot e & 0 & \ldots & 0 \\
0 & 0 & p_3^*\lambda \cdot e & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & p_s^*\lambda \cdot e \\
\end{pmatrix} R.$$

Assume $u^i(x^0_0, x^1_1, \ldots) > u^i(x^0_s(\lambda), x^1_s(\lambda), \ldots)$. We claim now that $(x^0_0, x^1_1, \ldots)$ can not be in the budget set corresponding to $(p^*\lambda, q^*\lambda)$. Assume on the contrary that

$$p_0^*\lambda (x^0_0 - \omega^0_0) \leq -q^*\lambda (\tilde{3}^i),$$

$$p_s^*\lambda (x^s_s - \omega^s_s) \leq V^*\lambda \tilde{3}^i, \forall s.$$

Let $\mu \in (0, 1)$ so that

$$x^i_s(\mu) = \mu x^i_s + (1 - \mu)x^*i(\lambda), \forall s \geq 0,$$

$$\tilde{3}^i(\mu) = \mu \tilde{3}^i + (1 - \mu)\tilde{3}^*i(\lambda).$$

We have

$$p_0^*\lambda (x^0_0(\mu) - \omega^0_0) = \mu p_0^*\lambda x^0_0 + (1 - \mu)p_0^*\lambda x^*i - \mu p_0^*\lambda \omega^0_0 - (1 - \mu)p_0^*\lambda \omega^*i,$$

and

$$q^*\lambda \tilde{3}^i(\mu) = \mu q^*\lambda \tilde{3}^i + (1 - \mu)q^*\lambda \tilde{3}^*i(\lambda),$$

that imply

$$p_0^*\lambda (x^0_0(\mu) - \omega^0_0) + q^*\lambda \tilde{3}^i(\mu) = \mu[p_0^*\lambda(x^0_0 - \omega^0_0) + q^*\lambda \tilde{3}^i] \leq 0,$$

since

$$p_0^\lambda (x^0_0 - \omega^0_0) + q^*\lambda \tilde{3}^i(\lambda) = 0.$$
Similarly, we obtain that
\[ p_s^\lambda(x_s^i(\mu) - \omega_0^i) \leq V_s^\lambda \hat{z}_i^i(\mu). \]

However, for \( \mu \) close to zero, \( x^i(\mu) \in \tilde{X}^i \), \( \hat{z}_i^i(\mu) \in \tilde{Z}^i \) and we have
\[
u^i(x^0_0(\mu), x^i_1(\mu), \ldots, x^i_S(\mu) \geq \nu^i(x^0_0(\mu), x^i_1(\mu), \ldots, x^i_S(\mu)) + (1 - \mu) \nu^i(x^0_0(\lambda), x^i_1(\lambda), \ldots, x^i_S(\lambda)),
\]
\[
u^i(x^0_0(\lambda), x^i_1(\lambda), \ldots, x^i_S(\lambda)).
\]
This contradicts with the fact that \((x^{*i}(\lambda), z^{*i}(\lambda), p^{*\lambda}, q^*(\lambda))\) is an equilibrium for \( \varepsilon(\lambda) \). Hence, \( \varepsilon \) has an equilibrium \((\bar{x}^i, \bar{z}^i, \bar{p}, \bar{q})\) where \((\bar{x}^i, \bar{z}^i, \bar{p}, \bar{q}) = (x^{*i}(\lambda), z^{*i}(\lambda), p^{*\lambda}, q^*(\lambda))\).

We have proved that there exists an incomplete financial equilibrium for the given economy with numeraire assets.

**References**