Long-Term Investment with Stochastic Interest and Inflation Rates: Incompleteness and Compensating Variation

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Abstract

We examine the long term investment problem, under stochastic interest and inflation rates and incompleteness. Four basic financial assets are available on the financial market: a money market account (the cash), a real consumption good, a financial stock index and a bond with constant maturity. This one corresponds to a nominal bond. In this incomplete framework, we provide the general solution of the expected utility maximization. This intertemporal optimization problem is solved by using the convex duality method, introduced by Cvitanic and Karatzas (1992). We determine also the optimal portfolio weights by using the method of dynamic programming based on the Hamilton-Jacobi-Bellmann approach. We compute the monetary loss from not having access to an indexed-inflation bond, in order to be hedged against the inflation risk, in particular for the logarithmic case.

Key words: portfolio optimization, stochastic interest rate, stochastic inflation, incompleteness, compensating variation.

JEL: C61, G11, G12.

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1 Introduction

Long term investment is very sensitive to inflation risk. In this framework, the prediction of real asset returns is a rather involved problem for long time horizons: are bonds more safe than equities? How inflation can modify portfolio performance? How can we hedge portfolios against inflation? How can we roll positions, since bonds with very long maturities are not available on financial markets? How can we model inflation rates and calibrate nominal interest rates?

In this paper, we solve the intertemporal portfolio optimization problem of an investor who searches to maximize the expected utility of her real wealth. We assume that both stochastic interest rate and inflation rate are stochastic and exhibit mean-reverting returns. We consider a financial market as in Chiarella et al. (2007), but with different basic tradable assets: we introduce bonds with constant durations. As discussed by Bajeux-Besnainou, Jordan and Portait (2001), the introduction of constant maturation bonds allows to obtain a Bond/Stock ratio which increases with time, when there exists no inflation. This nice property is in accordance with popular advice.

As in BJP (2001), we assume that the investor’s horizon exceeds the maturity of the cash asset and they introduce a continuous-time portfolio rebalancing. In that case, cash may be a money market security with a short maturity (one to six months) and may be no longer the common riskless asset in the standard theory. The portfolio is optimally rebalanced as a function of the remaining time and of current asset values. To model the multifactor term structure, we adopt the model introduced by Chiarella et al. (2007). Both indexed and non-indexed inflation bonds prices are based on exponential affine models, as introduced by Duffie and Kan (1996). They depend only on two factors: the real interest rate and the inflation rate. In this framework, both inflation-indexed bonds and nominal bonds can be priced.

Contrary to Mkaouar and Prigent (2008), we assume that inflation-indexed bonds are not available on the financial market, which implies that no perfect hedge exists against the inflation risk. We examine the consequences of the lack of inflation-indexed bonds since these latter ones are assumed to be no more available on the financial market. In that case, the investor cannot be perfectly hedged against the inflation risk. The financial market becomes incomplete. In this framework, two problems are studied: the first one is the resolution of the optimization problem. For this purpose, first we use the martingale approach introduced by Cvitanic and Karatzas (1992) to solve optimization problems when strategies must respect convex constraints. The idea is to complete the market with a fictitious asset, on which the investor cannot trade. Second, we use the dynamic programming approach based on Hamilton-Jacobi-Bellman (HJB) method, introduced for

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1 Inflation-indexed bonds are bonds the principal of which is indexed to inflation. They are introduced to cut out the inflation risk. The market of inflation-indexed bond has grown dramatically (1.5 trillion of the international debt market, in 2008). For instance, in France, there exists a bond indexed on inflation, called OATi, since December 1998.

2 Younger investors invest usually a higher fraction of their portfolio value in stocks than older investors. But, as mentioned by Samuelson (1963), this is not consistent with results of basic models of portfolio optimization. Canner, Mankiv and Weil (1997) note also that popular investment advice does not conform these results. Additionally, many empirical studies show that allocations between stocks, bonds and cash depend a lot on risk aversion. In particular, bond/stock ratios differ for conservative, moderate or aggressive investors. Bajeux-Besnainou, Jordan and Portait (2001), later on referred to as BJP (2001) address this inconsistency issue between mutual fund property and popular advice.

3 For simplicity, and in order to obtain analytical solutions, we assume that there is no transaction cost. Prices of these assets are also assumed to follow standard diffusion processes.
the first time by Merton (1971). A well-known property of optimal portfolio is the *mutual fund separation* theorem, which proves that a rational investor divides her investment between two assets: a riskless one and a risky mutual fund, the composition of which is the same, whatever the investor’s risk aversion. Cass and Stiglitz (1970) established conditions for two fund-separation, which holds for a large class of utility functions, such as the HARA functions. We show that, in our framework, this result is also true when examining dynamic portfolio strategies with incompleteness due to inflation.

Another consequence of the lack of inflation-indexed bonds is that the new optimal portfolio induces an utility loss when the same initial amount is invested. Several measures can be proposed to quantify the loss of not having access to inflation-indexed bonds. As shown in de Palma and Prigent (2008, 2009), we can introduce either a direct calculation of the utility loss, either a monetary loss based on the concept of *compensating variation*, which is linked to the notion of *certainty equivalence*. We use the latter one to examine the consequence of the lack of the inflation-indexed bonds.

The paper is organized as follows. Section 2 presents the financial market with its multi factor structure model that describes both the nominal and the inflation-indexed bonds. We determine the risk-neutral probabilities for the complete auxiliary markets, as introduced in Cvitanic and Karatzas (1992). Section 3 provides the solution of the optimization problem for quite general utility functions. We detail and illustrate the solution for the logarithmic, CRRA and CARA cases. Section 4 examines the consequences of the lack of inflation-indexed bonds, when these later ones are no more available on the financial market. Some of the proofs are gathered in the appendix.
2 The Financial Market

We adopt the same framework as in Chiarella et al. (2007). It is a generalization of Black and Scholes model and a variant of the Merton (1971) one state variable model. The model that we consider assumes normality of log returns. This assumption is not a too restrictive, when dealing with long term investment. This allows in particular to get explicit formulas.4

2.1 Asset dynamics

The market is assumed to be arbitrage-free and without friction. Financial transactions occur in continuous-time, along a time period \([0, T]\).

Five basic assets are available at any time on the market.

1. An instantaneously riskless money market fund, the \(\text{Cash}\), with a price denoted by \(C\).
2. A real consumption good \(C^I\).
3. A \(\text{Stock}\) index fund with a price \(S\).
4. A \(\text{Bond}\) fund \(B_D\) with constant duration \(D\), obtained by rolling continuously bonds throughout the investment period \([0, T]\). It is denoted by \(B_D\) which is a zero-coupon bond with maturity \((t + D)\) at time \(t\).
5. An Indexed-Inflation Bond (IIB) \(B_D^{IIB}\) indexed with constant duration \(D_i\), obtained by rolling continuously bonds throughout the investment period \([0, T]\). It is denoted by \(B_I\) which is a zero-coupon bond with maturity \((t + D_i)\) at time \(t\).

Thus, since \(D > d\), the riskless asset is a zero-coupon nominal bond \(B_T\) that matures at the investor’s horizon and which is replicated by a dynamic combination of \(C\), \(B_D\) and \(S\). Therefore, there exists a two fund Cass-Stiglitz separation with a synthetic riskless fund \(B_T\) replicated by \(C\) and \(B_D\) and a risky fund replicated by \(C\), \(B_D\) and \(S\). Since continuous-time rebalancing is allowed, financial markets can be assumed to be complete by introducing two sources of risk. In fact, as shown in Duffie and Huang (1985), such assumption of dynamic market completeness is allowed when contingent claims can be synthesized by continuous-time rebalancing.

4 Obviously, other models can be introduced and examined. For example, we can take account of stochastic market prices of risk, of labor income as in El Karoui and Jeanblanc (1998), for no traded asset such as in Lioui and Poncet (2001)... Nevertheless note that, even in complete markets, Monte Carlo simulations are often necessary to compute optimal portfolios as for example in Detemple, Garcia and Rindisbacher (2000) or in Cvitanic, Goukasian and Zapatero (2003).

5 As mentioned in BJP (2001), if inflation uncertainty is ignored, the interest rate risk means estate rate risk. The omission of inflation uncertainty may induce some problem when considering long horizons. Nevertheless, first some empirical evidence indicates, for example, that the US inflation volatility is smaller than real interest rate volatility (see Pennaci (1991) for example). Second, special long term bonds indexed on inflation are now available on some financial markets (for example, in US or UK). Finally, it is possible to diversify by investing in housing market.

6 Treasury Inflation Protected Securities (or TIPS) corresponds to the inflation-indexed bonds that are issued by the U.S. Treasury. These assets were first issued in 1997. The principal is adjusted to the Consumer Price Index, the commonly used measure of inflation. The coupon rate is constant, but generates a different amount of interest when multiplied by the inflation-adjusted principal, thus protecting the holder against inflation. TIPS are currently offered in 5-year, 10-year and 20-year maturities. 30-year TIPS are no longer offered.3

In addition to their value for a borrower who desires protection against inflation, TIPS can also be a useful information source for policy makers: By comparing a TIPS bond with a standard nominal Treasury bond across the same maturity dates, investors may calculate the bond market’s expected inflation rate by applying the Fisher equation.
To illustrate the results, we introduce a multidimensional Brownian motion:

\[ W = (W^r, W^i, W^I, W^S) \]

to describe the uncertainty of the asset returns. The correlation matrix is given by:

\[
\Sigma_c = \begin{bmatrix}
1 & \rho^{r,i} & \rho^{r,I} & \rho^{r,S} \\
\rho^{r,i} & 1 & 0 & \rho^{i,S} \\
\rho^{r,I} & 0 & 1 & \rho^{I,S} \\
\rho^{r,S} & \rho^{i,S} & \rho^{I,S} & 1
\end{bmatrix}
\]

Recall that the predictable compensator \( \langle W^a, W^b \rangle \) of the product \( W^a W^b \) satisfies:

\[ d\langle W^a, W^b \rangle_t = \rho^{a,b} dt. \]

### 2.1.1 Basic asset dynamics

1. We assume that the real interest rate \( r_t \) follows an Ornstein-Uhlenbeck process given by:

\[ dr_t = k_r(\overline{r} - r_t)dt + a_r dW^r_t, \quad (1) \]

where \( k_r, \overline{r} \) are positive constants.

2. We suppose that the expected rate of inflation \( \hat{i}_t \) follows an Ornstein-Uhlenbeck process given by:

\[ di_t = k_i(\overline{i} - \hat{i}_t)dt + a_i dW^i_t, \]

where \( k_i, \overline{i} \) are positive constants.

3. The nominal price \( I_t \) of the consumption good follows the dynamics:

\[ \frac{dI_t}{I_t} = i_t dt + \sigma_I dW^I_t. \]

4. The consumption good \( C^{I(r)}_t \) account satisfies:

\[ \frac{dC^{I(r)}_t}{C^{I(r)}_t} = r_t dt, \quad (2) \]

5. We denote \( C^{I(n)}_t \) the real money account given by: (nominal value)

\[ C^{I(n)}_t = C^{I(r)}_t \times I_t, \quad (3) \]

which gives the nominal value of the consumption good account.

6. The stock index satisfies: (nominal price)

\[ \frac{dS_t}{S_t} = \mu^S dt + \sigma^S dW^S_t, \quad (4) \]

Note that, for any real asset \( X^{(r)} \), we can deduce its nominal value asset denoted by \( X^{(n)} \) by using the following equality:

\[ X^{(n)} = X^{(r)} \times I. \]
Then, applying Yor’s lemma, we deduce:

\[
\frac{dX^{(n)}}{X^{(n)}} = \frac{dX^{(r)}}{X^{(r)}} + \frac{dI_t}{I_t} + \frac{1}{X^{(n)}} d\left[X^{(r)}, I\right].
\] (5)

### 2.1.2 The multi-factor model

Recall the determination of the multi-factor model for nominal and inflation-indexed bonds in Chiarella et al. (2007) and Mkaouar and Prigent (2008). The bond pricing formula is based on an exponential affine model, as introduced by Duffie and Kan (1996):

\[
B(r_t, i_t, t, T) = \exp \left[ -\alpha(T - t) - \beta_r(T - t) r_t - \beta_i(T - t) i_t \right],
\] (6)

where \(\alpha(\tau), \beta_r(\tau),\) and \(\beta_i(\tau)\) are determined by using the no-arbitrage condition. Due to the normalization, the coefficients satisfy the following terminal conditions at maturity:

\[
\alpha(0) = 0, \beta_r(0) = 0, \text{ and } \beta_i(0) = 0.
\] (7)

Applying Itô’s lemma to (6), the return of the nominal bond is given by:

\[
\frac{dB(r_t, i_t, t, T)}{B(r_t, i_t, t, T)} = \mu(t, \tau) dt - \beta_r(\tau) a_r \, dW^r_t - \beta_i(\tau) a_i \, dW^i_t,
\] (8)

We have:\^8

\[
\mu(t, \tau) = \alpha'(\tau) + \beta'_r(\tau) r_t + \beta'_i(\tau) i_t - \beta_r(\tau) k_r(\tau - r_t) - \beta_i(\tau) k_i(\tau - i_t)
\]

\[+ \frac{1}{2} \left[ \beta'^2_r(\tau) a_r^2 + \beta'^2_i(\tau) a_i^2 + 2 \beta_r(\tau) \beta_i(\tau) a_r a_i \rho^{r,i} \right],
\] (9)

where \(\tau = T - t\) denotes the remaining time to maturity.

The nominal yield is defined by the following relations:

\[
Y(t, T) = \frac{-\ln B(t, T)}{T - t} = \frac{\alpha(T - t)}{T - t} + \frac{\beta_r(T - t)}{T - t} r_t + \frac{\beta_i(T - t)}{T - t} i_t.
\] (10)

The instantaneous nominal interest rate \(R_t\) is defined as the instantaneous yield, which satisfies:

\[
R_t = \lim_{T\to t} Y(t, T).
\]

\^7 Denote by \(X = \mathcal{E}[Z]\) the Dade-Doleans exponential of the process \(Z\). It means that \(X\) satisfies the (SDE):

\[dX = X \cdot dZ.\]

Yor’s Formula: for any two processes \(X\) and \(Y\), we have:

\[\mathcal{E}[X] \cdot \mathcal{E}[Y] = \mathcal{E}[X + Y + [X, Y]],\]

where \([X, Y]\) denotes the quadratic covariation of \(X\) and \(Y\).

\^8 See Appendix A.
Applying this latter result to the yield formula (10), we deduce the expression:

\[ R_t = f'(0) + \beta'_t(0)r_t + \beta'_t(0)i_t, \tag{11} \]

where \( f' \) denotes the derivative of \( f \).

- The nominal money account is defined as the accumulation account:

\[ C(t) = \exp \left( \int_0^t R_s \, ds \right). \tag{12} \]

- Let \( B^I_t \) denote the nominal price of the (zero-coupon) inflation indexed bond (IIB) that is issued at time 0 and with maturity \( T \). At maturity, its payout must be equal to the price index \( I_T \). Thus, we get:

\[ B^I(T, T) = I_T. \tag{13} \]

- The real Indexed-Inflation bond with maturity \( T \) is defined as follows:

\[ B^{I(r)}(t, T) = \frac{B^I(t, T)}{I_t}. \]

It corresponds to the normalized IIB with respect to the corresponding price index. According to (13), we have \( B^{I(r)}(T, T) = 1 \). This property means that the real bond provides one unit of consumption good at time \( T \). We assume that the real bond is only affected by one factor, which is the instantaneous real interest rate \( r_t \). We assume also that it follows the Duffie and Kan dynamics:

\[ B^{I(r)}(t, T) = \exp \left[ -\alpha_B^{I(r)}(T - t) - \beta_B^{I(r)}(T - t)r_t \right], \tag{14} \]

where the Duffie-Kan coefficients \( \alpha_B \) and \( \beta_B^{I(r)} \) are deduced from the non-arbitrage conditions, as shown in what follows. Note that we have:

\[ \alpha_B(0) = 0, \text{ and } \beta_B^{I(r)} = 0. \tag{15} \]

The assumption (14) concerning the real bond determines the dynamics for \( B^I(t, T) \). Since the real yield is the constant interest rate of the real bond, we deduce:

\[ Y_r(t, T) = \frac{-\ln \left[ B^{I(r)}(t, T) \right]}{T - t} = \frac{\alpha_B^I(T - t)}{T - t} + \frac{\beta_B^{I(r)}(T - t)}{T - t}r_t. \]

The return of the IIB is computed by applying Itô’s Lemma to the real bond price formula (14). We get:

\[ \frac{dB^{I(r)}(r_t, t, T)}{B^{I(r)}(r_t, t, T)} = \mu^{B^{I(r)}}(t, T - t)dt - \beta_{B^{I(r)}}(T - t)a_r dW^r_t, \tag{16} \]

where:

\[ \mu^{B^{I(r)}}(t, \tau) = \alpha_{B^{I}}(\tau) + \beta'_{B^{I(r)}}(\tau)r_t - \beta_{B^{I(r)}}(\tau)k_r(\tau - r_t) + \frac{1}{2}\beta_{B^{I(r)}}^2(\tau)a_r^2. \tag{17} \]

Next applying Itô’s Lemma to the expression for the IIB,

\[ \frac{dB^{I(n)}(r_t, i_t, t, T)}{B^{I(n)}(r_t, i_t, t, T)} = \mu^{B^{I(n)}}(t, T - t)dt - \beta_{B^{I(n)}}(T - t)a_r dW^r_t + \sigma_I dW^I_t, \]

where: \( \mu^{B^{I(n)}} \) is the drift term, \( \beta_{B^{I(n)}} \) is the diffusion term, and \( \sigma_I \) is the volatility term.
where:
\[ \mu^{B^{(n)}(t, T - t)} = \mu^{B^{(r)}(t, T - t)} + i_t - \beta_{B^{(r)}} (T - t) a_r \sigma_1 \rho^{r,i}. \]  

The return on the real money account \( C^{I(n)} \) can be deduced from (3):
\[ \frac{dC^{I(n)}}{C^{I(n)}} = (r_t + i_t) dt + \sigma_I dW_I^t. \]

In order to obtain the bond price, we employ the standard no-arbitrage argument as in Chiarella et al. (2007).

The financial market is complete. Therefore, there exists a unique risk-neutral probability \( Q \) associated to four market premia, \( \lambda_r, \lambda_i, \lambda_I \) and \( \lambda_S \) which density \( \eta \) with respect to the initial probability \( P \) is given by:
\[ \eta_t = \mathbb{E} \left[ \frac{dQ}{dP} | \mathcal{F}_t \right] = \mathcal{E} [-M]. \]  

Thus, we have also:
\[ \eta_t = \exp [-M - \Lambda t] \]
where
\[ M = \int_0^t \lambda_r dW^r + \int_0^t \lambda_i dW^i + \int_0^t \lambda_I dW^I + \int_0^t \lambda_S dW^S, \]
and
\[ \Lambda = \frac{1}{2} (\lambda_r^2 + \lambda_i^2 + \lambda_I^2 + \lambda_S^2) + \lambda_r \lambda_i \rho^{r,i} + \lambda_r \lambda_I \rho^{r,I} + \lambda_r \lambda_S \rho^{r,S} + \lambda_i \lambda_I \rho^{i,I} + \lambda_I \lambda_S \rho^{I,S}. \]

Each of the four basic assets \( B, B^{(n)}, C^{I(n)} \) and \( S \) must satisfy the following condition: when they are discounted by the nominal money account \( C \), they must be martingales with respect to the risk-neutral probability \( Q \). This is equivalent to the fact that when are multiplied by the Radon-Nikodym density \( \eta \) and divided by \( C \), they must be martingales with respect to the historical probability \( P \). It is equivalent to the fact that their bounded variation components are equal to 0. This latter condition implies four equalities detailed in appendix A.

Then we can determine the risk factors \( \lambda_r, \lambda_i, \lambda_I \) and \( \lambda_S \). Indeed, introduce the matrix \( \Gamma \) equal to:
\[
\begin{bmatrix}
-a_r \beta^{r,i} & -a_r \beta^{r,i} & -a_r \beta^{r,i} & -a_r \beta^{r,i} \\
\sigma_1 \rho^{r,i} - a_r \beta^{B^{(r)}} & \sigma_1 \rho^{r,i} - a_r \beta^{B^{(r)}} & \sigma_1 \rho^{r,I} & \sigma_1 \rho^{r,i} - a_r \beta^{B^{(r)}} \\
\sigma_1 \rho^{r,i} & 0 & \sigma_I - a_r \beta^{B^{(r)}} \rho^{r,i} & \sigma_I - a_r \beta^{B^{(r)}} \rho^{r,I} \\
0 & \sigma_I & \sigma_I & \sigma_I \\
\end{bmatrix}
\]

We have:
\[
\begin{bmatrix}
\theta^B \\
\theta^{B^{(n)}} \\
\theta^{C^{I(n)}} \\
\theta^S \\
\end{bmatrix} = \Gamma \begin{bmatrix}
\lambda_r \\
\lambda_i \\
\lambda_I \\
\lambda_S \\
\end{bmatrix}, \quad \text{thus:} \quad \begin{bmatrix}
\lambda_r \\
\lambda_i \\
\lambda_I \\
\lambda_S \\
\end{bmatrix} = \Gamma^{-1} \begin{bmatrix}
\theta^B \\
\theta^{B^{(n)}} \\
\theta^{C^{I(n)}} \\
\theta^S \\
\end{bmatrix}.
\]
Therefore, we deduce:

\[
\begin{align*}
\lambda_r &= -x_B(t)\beta_r(D)a_r - x_B(t)\beta_B(D)a_r \\
\lambda_i &= -x_B(t)\beta_i(D)a_i \\
\lambda_I &= x_C(t)\sigma_I + x_B(t)\sigma_I - \sigma_I \\
\lambda_S &= x_S(t)\sigma_S
\end{align*}
\]

3 Optimal portfolios

In what follows, we assume that the investor can invest only on four basic assets: the Indexed-Inflation Bond (IIB) is not traded on the financial market. Thus, this latter one is incomplete.

Portfolio weights are respectively denoted by \(x_C, x_B, x_{C_i}, \) and \(x_S\). The nominal portfolio value at time \(t\) is denoted by \(V_t^{(n)}\). We have: \(x_C = 1 - x_B - x_{C_i} - x_S\). Therefore, the portfolio value \(V\) follows the dynamics:

\[
\begin{align*}
\frac{dV_t^{(n)}}{V_t^{(n)}} &= [R_t + x_B(t)\theta^B + x_C(t)\theta^C + x_S(t)\theta^S]dt \\
&+ x_B(t) [-\beta_r(D)a_r dW_t^r - \beta_i(D)a_i dW_t^i] + x_C(t) [\sigma_I dW_t^I] + x_S(t) [\sigma_S dW_t^S].
\end{align*}
\]

Denote by \(\Sigma\) and \(\bar{\Sigma}\) the asset return matrices, respectively of the four and three assets (i.e. respectively \((B, B^I(n), C^I(n), S)\) and \((B, C^I(n), S)\)) in nominal values. They are respectively given by:

\[
\Sigma = \begin{bmatrix} -a_r\beta_r & -a_i\beta_i & 0 & 0 \\ -a_r\beta_I & 0 & \sigma_I & 0 \\ 0 & 0 & \sigma_I & 0 \\ 0 & 0 & 0 & \sigma_S \end{bmatrix} \quad \text{and} \quad \bar{\Sigma} = \begin{bmatrix} -a_r\beta_r & -a_i\beta_i & 0 & 0 \\ 0 & 0 & \sigma_I & 0 \\ 0 & 0 & 0 & \sigma_S \end{bmatrix}.
\]

Introduce the matrix \(\bar{\Gamma}\) equal to:

\[
\bar{\Gamma} = \begin{bmatrix} -a_i\beta_r \rho_r \rho_i - a_r\beta_r & -a_r\beta_i & 0 & 0 \\ 0 & \sigma_I \rho_r \rho_i - a_r\beta_r \rho_r \rho_i - a_r\beta_i & 0 & 0 \\ 0 & 0 & \sigma_I \rho_r \rho_i - a_r\beta_r \rho_r \rho_i - a_r\beta_i & 0 \\ 0 & 0 & 0 & \sigma_S \end{bmatrix}.
\]

Denote \(\bar{\theta} = t(\theta^B, \theta^C, \theta^S)\) and \(\bar{x} = t(x_B, x_{C_i}, x_S)\).

The investor’s preferences is described by utility function \(U\), which embeds her risk aversion. We consider an investor with an initial capital denoted by \(V_0\). She is assumed to maximize the expected utility over the time horizon \(T\), defined on the real portfolio value \(V = V_t^{(n)}/I\).

Note that we have:

\[
d\left(\frac{1}{I}\right)_t = \left(\frac{1}{I}\right)_t \left[ (-i_t + \sigma_t^2) dt - \sigma_t dW_t^I \right].
\]

Therefore, we deduce:

\[
\frac{dV}{V} = \frac{d\left(\frac{V_t^{(n)}}{I}\right)}{V_t^{(n)}} = \frac{dV_t^{(n)}}{V_t^{(n)}} + \frac{d\left(\frac{1}{I}\right)}{I} + \frac{1}{V} d\left[ V_t^{(n)} \left(\frac{1}{I}\right)_t \right],
\]

\[9\]
then:
\[
\frac{1}{V} \frac{dV}{V} = \frac{1}{T} [-\sigma_I \left( x_B(t) \left( -\beta_r(D)a_r \rho^{r,r_i} \right) + x_{C_i}(t) \sigma_I + x_S(t) \sigma^S \rho^{I,S} \right)] dt.
\]

Finally, we get:
\[
\frac{dV}{V} = \left[ R_t + x_B(t) \theta^B + x_{C_i}(t) \theta^C + x_S(t) \theta^S \right] dt
- \sigma_I \left[ x_B(t) \left( -\beta_r(D)a_r \rho^{r,r_i} \right) + x_{C_i}(t) \sigma_I + x_S(t) \sigma^S \rho^{I,S} \right] dt
+ x_B(t) \left[ -\beta_r(D)a_r dW_t^r - \beta_i(D)a_i dW_t^i \right] + x_{C_i}(t) \left[ \sigma_I dW_t^I \right] + x_S(t) \left[ \sigma^S dW_t^S \right]
+ \left[ (-\sigma^2_I) dt - \sigma_I dW_t^I \right]
\]

This expression is equivalent to:
\[
\frac{dV_t}{V_t} = \left[ \left( R_t - i_t + \sigma^2_t \right) + \tilde{x}^T(t) \begin{pmatrix} \tilde{\theta}_t - \sigma_I \tilde{\Gamma}_t D^I \end{pmatrix} \right] dt + \tilde{x}^T(t) \tilde{\Sigma}_t dW_t - \sigma_I dW_t^I,
\]
where \( D^I = \iota(0,0,1,0) \).

Thus, her optimal portfolio weights are the solutions of the following problem: (recall that \( x_C = 1 - x_B - x_{C_i} - x_S \) and that \( V \) denotes the real portfolio value)
\[
\max_{x_B,x_{C_i},x_S} E \left[ U(V_T) \right].
\]

For different utility functions, we determine the optimal portfolio. We show how it depends on the investor risk aversion.\(^9\) We detail the solutions for HARA and CARA utility functions.

### 3.1 General result (Martingale approach)

In the continuous-time setting, Cvitanic and Karatzas (1992) study the stochastic dynamic control problem, which corresponds to the maximization of the expected utility of consumption and terminal value. The set of constraints is supposed to be a given closed and convex subset of \( \mathbb{R}^d \). The idea is to embed the constrained problem in a family of unconstrained ones. Then, we search for an element of this family which satisfies the given constraints. Such results can be applied in particular to incompleteness and when there is no short-selling. This method is based on martingale theory, duality theory and convex analysis.

Let \( K \) be a set of constraints which are assumed to be such that the set \( K \) is a nonempty, closed and convex subset of \( \mathbb{R}^d \).

Denote by \( \delta \) the support function of the convex set \(-K\), defined on \( \mathbb{R}^d \) and with values in \( \mathbb{R}^d \cup \{+\infty\} \):
\[
\delta(x) = \delta(x,K) = \sup_{w \in K} (-\iota w \cdot x).
\]

The function \( \delta \) is a closed, positively homogeneous, proper convex function on \( \mathbb{R}^d \) (see for example, Rockafellar (1970) for details about these notions).

\(^9\)See Gollier (2001) for definitions and main properties of utility functions.
The effective domain of the function $\delta$ is the set $\tilde{K}$ defined by:

$$\tilde{K} = \{ x \in \mathbb{R}^d, \delta(x) < \infty \},$$

$$= \{ x \in \mathbb{R}^d, \exists \beta \in \mathbb{R}, -t^w.x \leq \beta, \forall w \in K \}.$$

The set $\tilde{K}$ is a convex cone, called the “barrier cone” of $-K$.

For the incomplete market case, we get:

$$K = \{ x \in \mathbb{R}^d, x_i = 0, \forall i = m + 1, \ldots, d \}, \text{ for some } m \in \{1, \ldots, d-1\}.$$

Then:

$$\delta(x) = \begin{cases} 0, & x_1 = \ldots = x_m = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

and

$$\tilde{K} = \{ x \in \mathbb{R}^d, x_i = 0, \forall i = 1, \ldots, m \}.$$

This case corresponds to an incomplete financial market driven by a standard multidimensional Brownian motion $W$, which contains $m$ stocks with $m < d$.

In our model, we get the following results for $d = 4, m = 3$.

The financial market is complete when the four basic assets are traded, in particular the indexed-inflation bond. Thus, there exists one and only one risk-neutral probability $Q$, which can be defined as follows.\(^{10}\) Denote the relative risk process $N$ by:

$$N(t) = \Gamma_f (t)^{-1} \theta(t),$$

with:

$$\mathbb{E} \left[ \int_0^t N(s) \Sigma_c N(s) ds \right] < \infty, \forall t \in [0, T].$$

The exponential local martingale $\eta$ is given by:

$$\eta(t) = \exp \left[ -\frac{1}{2} \int_0^t N(s) \Sigma_c N(s) ds - \int_0^t N(s) dW_s \right],$$

is the Radon-Nikodym density of the risk-neutral probability $Q$ w.r.t. the probability $P$.

Denote $\mathcal{D}$ the discount factor:

$$D(t) = \exp \left[ -\int_0^t R(s) ds \right] = \frac{1}{C_t^{1/2}}.$$

The utility functions satisfies all the standard assumptions. In particular, the conjugate function of an utility function $U$ (Legendre-Fenchel transform) is still denoted by $\hat{U}$:

$$\hat{U}(y) = \max_{v > 0} [U(y) - vy], \quad y > 0, \tag{22}$$

and the inverse of the derivative marginal utility $U'$ is denoted by $J$.

\(^{10}\)See also Relation (19).
- **Step 1. The constrained optimization problem.**

The investor searches to maximize

$$E[U(V_T)],$$

on the set $\mathcal{A}_K(v_0)$ of all admissible strategies $x$ satisfying usual assumptions:

$$\mathcal{A}_K(v_0) = \{x, x(\omega, t) \in K, \text{ for } \mathbb{P} \otimes dt \text{ a.s. } (\omega, t)\}.$$

The new value function $J_K$ is defined by:

$$J_K = \sup_{x \in \mathcal{A}_K(v_0)} E[U(V_T)].$$

Note that, when $K = \mathbb{R}^d$ (no constraint), recall that the (real) solution is given by:

$$V_T^* = J(\lambda^* I_T \kappa(T)),$$

where $\kappa(t) = D(t) \eta(t)$.

The Lagrange parameter $\lambda^*$ is determined from the budget equation. Its existence is deduced from assumptions on utility function, since the function $F$, defined by

$$F(y) = \mathbb{E}_\mathbb{P}[I_T \kappa(T) J(y I_T \kappa(T))],$$

is continuous and non-increasing on $[0, a]$ where $a = \inf \{y | F(y) = 0\}$. It satisfies also:

$$\lim_{y \to 0} F(y) = +\infty; \lim_{y \to +\infty} F(y) = 0.$$

Then, the function $F$ has an inverse $F^{-1}$.

Under assumptions for utility function $U$, there exists an optimal strategy $x^* \in \mathcal{A}(v_0)$ such that:

$$\mathcal{J}(x^*, v_0) = \max_{x \in \mathcal{A}(v_0)} \mathcal{J}(x, v_0) \quad (23)$$

where

$$\mathcal{J}(x, v_0) = \mathbb{E}_\mathbb{P}[U(V_T^*)]. \quad (24)$$

Using previous function $F$ and the inverse functions $J$ of marginal utility functions $U'$, we have:

$$V_T^* = J(F^{-1}(v_0) I_T \kappa(T)),$$

and the optimal weighting $x^*$ is deduced from the martingale representation.

- **Step 2. The auxiliary unconstrained optimization problem.**

Cvitanic and Karatzas (1992) introduce a family of unconstrained optimization problems which embeds the constrained problem. For this purpose:

- Consider the space $\mathcal{H}$ of $\mathcal{F}_t$-progressively measurable processes $(v_t)_t$ with values in $\mathbb{R}^d$ and such that:

$$||v||^2 = \mathbb{E} \left[ \int_0^T ||v_t||^2 dt \right] < \infty.$$
• Introduce the class $\mathcal{D}$ of processes such that: $\mathcal{D} = \left\{ \nu \in \mathcal{H}, \int_0^T \delta(v_t) \, dt \leq \infty \right\}$, where $\delta(.)$ is the support function of the set of constraints $K$. Note that:

$$\nu \in \mathcal{D} \iff \nu(\omega, t) \in \tilde{K}, \text{ for } \mathbb{P} \otimes dt \text{ a.s. } (\omega, t),$$

(25)

where $\tilde{K}$ is the barrier cone of $K$.

For any given $\nu \in \mathcal{D}$, consider a new financial market $\mathcal{M}$, with one bond and $d$ stocks:

$$dC_t^{(v)} = C_t^{(v)} \left[ R(t) + \delta(v_t) \right] \, dt,$$

(26)

$$dS_{i,t}^{(v)} = S_{i,t}^{(v)} \left[ \mu_i(t) + v_i(t) + \delta(v_t) + \sum_{j=1}^d \gamma_{i,j}(t) dW_{j,t} \right].$$

(27)

$\Sigma_f$ corresponds to the matrix $[\gamma_{i,j}]_{i,j}$.

Associated to the process $\nu \in \mathcal{D}$, denote also:

* The relative risk process $N^{(v)}$ by:

$$N^{(v)}(t) = \Gamma_f(t)^{-1} \left[ \mu(t) + v(t) + \delta(v_t) I_{\mathbb{R}d} - (R(t) + \delta(v_t)) I_{\mathbb{R}d} \right] = N(t) + \Gamma_f^{-1} \nu(t).$$

(28)

* The exponential local martingale $\eta^{(v)}$:

$$\eta^{(v)}(t) = \exp \left[ -\frac{1}{2} \int_0^t N^{(v)}(s) \Sigma \eta^{(v)}(s) ds - \int_0^t N^{(v)}(s) dW_s \right],$$

(29)

is the Radon-Nikodym density of the risk-neutral probability $Q^{(v)}$ w.r.t. the probability $\mathbb{P}$.

* Denote $D^{(v)}$ the discount factor:

$$D^{(v)}(t) = \exp \left[ -\int_0^t (R(s) + \delta(v_s)) \, ds \right].$$

(30)

* Denote also $\kappa^{(v)}$ their product: $\kappa^{(v)}(t) = \left[ D^{(v)}(t) \eta^{(v)}(t) \right]$.

* Consider the new set of admissible strategies:

- The nominal wealth process $V^{(n)(v)}$ satisfies:

$$dV_t^{(n)(v)} = \left[ R(t) + \int_t^T x(t) \, dt \right] V_t^{(n)(v)} + V_t^{(n)(v)} \left[ \int_t^T \Sigma \, dW_t^{(v)} \right],$$

(31)

where $W_t^{(v)} = W_t + \int_0^t N^{(v)}(s) \, ds$ is a Brownian motion under the risk-neutral probability $Q^{(v)}$.  

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- The investor searches to maximize (through the strategy \( x^{(v)} \)):
  \[
  \mathbb{E} \left[ U(V_T^{(v)}/I_T) \right],
  \tag{32}
  \]
on the set \( \mathcal{A}^{(v)} \) of all admissible strategies \( x^{(v)} \):
  \[
  \mathcal{A}^{(v)}(v_0) = \left\{ x^{(v)}, x^{(v)}(\omega, t) \in K, \text{ for } \mathbb{P} \otimes dt \text{ a.s. } (\omega, t) \right\}.
  \tag{33}
  \]

The value function \( J \) is still defined by:
  \[
  J^{(v)} = \sup_{x^{(v)} \in \mathcal{A}^{(v)}} \mathbb{E} \left[ U(V_T^{(v)}) \right].
  \tag{34}
  \]

The Lagrange parameter \( \lambda^{(v)*} \) is determined from the budget equation. Its existence is from assumptions on the utility function: indeed, the function \( F^{(v)} \), defined by
  \[
  F^{(v)}(y) = \mathbb{E}_\mathbb{P} \left[ I_{T^K(v)}(T) J \left( y I_{T^K(v)}(T) \right) \right],
  \tag{35}
  \]
has an inverse \( F^{(v)^{-1}} \).

**Proposition 1** Under assumptions for utility functions \( U \), there exists an optimal strategy \( x^{(v)*} \in \mathcal{A}^{(v)}(v_0) \) such that:
  \[
  V_T^{(v)*} = J(F^{(v)^{-1}}(v_0)I_{T^K(v)}(T)),
  \]
and the optimal weighting \( x^{(v)*} \) is deduced from the martingale representation.

Using previous function \( F \) and the inverse functions \( J \) of marginal utility functions \( U' \), we have:
  \[
  V_T^{(v)*} = J(F^{(v)^{-1}}(v_0)I_{T^K(v)}(T)).
  \]

- **Step 3. Equivalent optimization conditions.**

**Proposition 2** Suppose that for some process \( \lambda \in \mathcal{D}' \), we have:
  \[
  x_t^{(\lambda)} \in K, \\ \delta (\lambda(\omega, t)) + t x_t^{(\lambda)} \lambda(\omega, t) = 0.
  \tag{36}
  \]

Then the strategy \( x^{(\lambda)*} \) belongs to \( \mathcal{A}^{(\lambda)}(v_0) \) and is optimal for the constrained optimization problem in the original market.

- Consider a solution \( x_K^* \) of the constrained optimization problem (A):
  \[
  \sup_{x_K \in \mathcal{A}_K(v_0)} \mathbb{E}[U(V_K,T)].
  \]
Cvitanic and Karatzas (1992) characterize the solution of Problem (A) by using the following conditions for a given process \( x_t \) in the class \( \mathcal{D} \):

* (B) Financibility of \( x^{(\lambda)}, V_T^{(\lambda)} \):

There exists a portfolio process \( x^{(\lambda)} \) such that \( x^{(\lambda)} \in \mathcal{A}_K \) and:

\[
x_t^{(\lambda)}(\omega, t) \in K, \quad \delta(\lambda(\omega, t)) + \int_t^T x_t^{(\lambda)} \lambda(\omega, t) dt = 0.
\]

\[
V^{(v_0, x^{(\lambda)})}(\omega, t) = V^{(\lambda)}(\omega, t) \text{ for } \mathbb{P} \otimes dt \text{ a.s. } (\omega, t).
\]

* (C) Minimality of \( \lambda \). For every \( \nu \in \mathcal{D} \), we have:

\[
\mathbb{E} \left[ U(V_T^{(\lambda)}) \right] \leq \mathbb{E} \left[ U(V_T^{(\nu)}) \right].
\]

* (D) Dual optimality of \( \lambda \). For every \( \nu \in \mathcal{D} \), we have:

\[
\mathbb{E}_p \left[ K^{(\lambda)}(T) I_T \bar{J} \left( y I_T K^{(\lambda)}(T) \right) \right] 
\leq \mathbb{E}_p \left[ K^{(\nu)}(T) I_T \bar{J} \left( y I_T K^{(\lambda)}(T) \right) \right]
\]

* (E) Parsimony of \( \lambda \). For every \( \nu \in \mathcal{D} \), we have:

\[
\mathbb{E}_p \left[ K^{(\nu)}(T) I_T V_T^{(\lambda)} \right] \leq v_0.
\]

**Theorem 3** (Equivalence of conditions) Conditions (B)-(E) are equivalent and imply property (A) with

\[
x^*_K = x^{(\lambda)}.
\]

Additionally, assumptions on utility function imply the existence of \( \lambda \in \mathcal{D} \) which satisfies conditions (B)-(E) with \( x^*_K = x^{(\lambda)} \) under the following assumptions:

(i) The utility function satisfies:

\[
v \rightarrow v U''(v) \text{ is nondecreasing on } (0, \infty)
\]

and for some \( \alpha \in (0, 1), \gamma \in (1, \infty) \), we have:

\[
\alpha U''(v) \geq U''(\gamma v), \text{ for all } v > 0.
\]

From the previous result, we are led again to the dual stochastic control problem:

\[
\mathcal{J}(y) = \inf_{\nu \in \mathcal{D}} \mathbb{E} \left[ \bar{J} \left( y I_T K^{(\nu)}(T) \right) \right].
\]

**Proposition 4** (Existence of a solution of the constrained optimization) Under all previous assumptions on the utility functions, there exists an optimal solution \( x^*_K \) for the constrained portfolio.
3.1.1 Logarithmic utility case

Assume that \( U(x) = \ln x \). Then, we have: for every \( \nu \in \mathcal{D} \),

\[
V^{(\nu)}(T) = \frac{V_0}{I_{TR^{(\nu)}}(T)}. \tag{41}
\]

Note that \( \mathcal{D} = \mathcal{D}' \). Thus:

\[
\mathbb{E}_\mathbb{P} \left[ \hat{U} \left( F^{(\nu)^{-1}}(V_0)I_{TR^{(\nu)}}(T) \right) \right] = -1 - \ln \left[ F^{(\nu)^{-1}}(V_0) \right] + \mathbb{E}_\mathbb{P} \left[ \ln \frac{1}{I_{TR^{(\nu)}}(T)} \right].
\]

Moreover, we have:

\[
\mathbb{E}_\mathbb{P} \left[ \ln \frac{1}{I_{K^{(\nu)}}(t)} \right] = \mathbb{E}_\mathbb{P} \left[ \int_0^t \left( r(s) + \alpha'(0) + \frac{1}{2} \sigma^2_t + \delta(\nu(s)) \right) \left( N(s) + \Gamma^{-1}_f(s) v(s) \right) \Sigma_c \left[ N(s) + \Gamma^{-1}_f(s) v(s) \right] ds \right].
\]

Therefore, the optimization problem is equivalent to a pointwise minimization of the convex function

\[
\delta(x) + \frac{1}{2} \left[ N(t) + \Gamma^{-1}_f x \right] \Sigma_c \left[ N(t) + \Gamma^{-1}_f x \right],
\]

over \( x \in \bar{K} \), for every \( t \in [0,T] \).

Thus the process \( \lambda \) is determined by:

\[
\lambda(t) = \arg \min_{x \in \bar{K}} \left[ 2\delta(x) + t \left[ N(t) + \Gamma^{-1}_f x \right] \Sigma_c \left[ N(t) + \Gamma^{-1}_f (t)x \right] \right]. \tag{42}
\]

Finally, we deduce:

\[
x_K(t) = \left[ \Gamma_f(t)\Gamma_f(t)^{-1} \right] \left[ \lambda(t) + \mu(t) - R(t)1 \right].
\]

For the incomplete cases, the constraint sets \( K \) is such that \( \delta(.) = 0 \) on \( \bar{K} \). Thus, for the logarithmic case, the problem of determining the process \( \lambda \in \mathcal{D} \) reduces to that of minimizing pointwise a simple quadratic form, over \( \bar{K} \):

\[
\lambda(t) = \arg \min_{x \in \bar{K}} \left[ t \left[ N(t) + \Gamma^{-1}_f x \right] \Sigma_c \left[ N(t) + \Gamma^{-1}_f (t)x \right] \right]. \tag{43}
\]

For the unconstrained case, \( \lambda(t) = 0 \), we recover the standard solution. The optimal weights are given by:

\[
x_K(t) = \left[ \Gamma_f(t)\Gamma_f(t)^{-1} \right] \left[ \theta_t \right].
\]

Therefore, for the logarithmic case, we get explicit formulas:

For the incomplete case, set:

\[
\Gamma_f(t) = \begin{bmatrix} \hat{\Gamma}_f(t) \\ \rho(t) \end{bmatrix},
\]

\[\text{Note that Cvitanic and Karatzas (1992) provide also a general result for other utility functions but only for the deterministic case (the drift and volatility coefficients are constant).} \]
where $\tilde{\Gamma}_f(\omega, t)$ is an $(m \times d)$ matrix of full rank and $\rho(\omega, t)$ is an $(n \times d)$ matrix with orthogonal rows that span the kernel of $\Gamma_f(\omega, t)$ (w.r.t. $\Sigma_c$), for every $(\omega, t)$ (we have: $\dagger \rho(\omega, t) \Sigma_c \rho(\omega, t) = I_n$ and $\tilde{\Gamma}_f(\omega, t), \Sigma_c$). $\rho(\omega, t) = 0$ and $n = d - m$).

Then, set:

$$
\begin{align*}
\tilde{m}(t) &= \dagger (\mu_1(t), \ldots, \mu_m(t)), \\
a(t) &= \dagger (\mu_{m+1}(t), \ldots, \mu_d(t)), \\
\tilde{N}(t) &= \dagger \tilde{\Gamma}_f(t) \left( \tilde{\Gamma}_f(t)^\dagger \tilde{\Gamma}_f(t) \right)^{-1} [\tilde{m}(t) - R(t)I_{Rm}].
\end{align*}
$$

We have:

$$
N(t) = \tilde{N}(t) + \dagger \rho(t) [a(t) - R(t)I_{Rn}].
$$

Note that for all $v \in \tilde{K}$, $v$ has the following type:

$$
\nu(t) = \begin{bmatrix} 0_m \\ x \end{bmatrix},
$$

Denote:

$$
P = \Gamma_f^{-1t}(0, 1, 0, 0).
$$

We deduce:

$$
\dagger \left[ N(t) + \Gamma_f^{-1} \nu \right] \Sigma_c \left[ N(t) + \Gamma_f^{-1}(t) \nu \right] = \dagger [N(t)] \Sigma_c N(t) + 2x \dagger [N(t)] \Sigma_c P + x^2 \dagger \left[ P \right] \Sigma_c P
$$

Thus, the minimization is achieved by the random vector $v$ corresponding to the solution $x$ of:

$$
\frac{\partial}{\partial x} \left( \dagger [N(t)] \Sigma_c N(t) + 2x \dagger [N(t)] \Sigma_c P + x^2 \dagger \left[ P \right] \Sigma_c P \right) = 0.
$$

This solution is given by:

$$
x^* = \dagger \frac{[N(t)] \Sigma_c P}{\dagger \left[ P \right] \Sigma_c P}.
$$

Then, we deduce that the optimal process $\nu$, denoted by $\lambda$, is equal to:

$$
\lambda(t) = \begin{bmatrix} 0 \\ l(t) \\ 0 \\ 0 \end{bmatrix}, \quad (44)
$$

where

$$
l(t) = -\dagger \frac{[N(t)] \Sigma_c P}{\dagger \left[ P \right] \Sigma_c P}.
$$

Using the previous general result, we deduce:
Proposition 5  For the logarithmic case, the optimal portfolio value, without investment on the indexed-inflation bond, is given by:

\[ V^\lambda(T) = \frac{V_0}{I_r^\lambda(T)} \text{ with } \lambda(t) = \begin{bmatrix} 0 \\ l(t) \\ 0 \end{bmatrix} \text{ where } l(t) = -\frac{t}{t}[N(t)]\Sigma_eP. \quad (45) \]

The optimal weights satisfy: \( x_{B_i} = 0 \) and

\[ \begin{bmatrix} x_B \\ x_{C_i} \\ x_S \end{bmatrix} = (\tilde{\Gamma}(t)\tilde{\Gamma}(t))^{-1} \begin{bmatrix} \theta^{B_i} \\ \theta^{C_i} \\ \theta^S \end{bmatrix}. \quad (46) \]

Proof.  We apply the previous general results to the four basic assets. In particular, we set:

\[ \tilde{\Gamma}_f = \tilde{\Gamma} = \begin{bmatrix} -a_i\beta_i\rho^{i,s} - a_r\beta_r - a_i\beta_i + a_r\beta_r\rho^{r,i} & -a_i\beta_i \rho^{i,s} - a_r\beta_r \rho^{r,s} \\ \sigma_f \rho^{i,s} & 0 \\ \sigma_s \rho^{i,s} & \sigma_s \rho^{r,s} & \sigma_s \end{bmatrix}, \]

We complete the market by a fifth (fictional) asset (here, the second asset, which is a surrogate to the indexed-inflation bond).

Then, the relative risk process \( N^{(\lambda)} \) satisfies:

\[ N^{(\lambda)}(t) = N(t) + \Gamma_f^{-1}\lambda(t), \text{ with } N(t) = \Gamma_f^{-1}\theta(t), \quad (47) \]

which leads to the following relation:

\[ N^{(\lambda)}(t) = \Gamma_f^{-1}(\theta^{B_i}, \theta^{B_i} + \lambda(t), \theta^{C_i}, \theta^S). \quad (48) \]

The local martingale exponential \( \eta^{(\lambda)} \) satisfies:

\[ \eta^{(\lambda)}(t) = \exp \left[ -\frac{1}{2} \int_0^t \left[ N(s) + \Gamma_f^{-1}(s)\sigma_e \left[ N(s) + \Gamma_f^{-1}(s)\sigma_e(s) \right] ds - \int_0^t N^{(\lambda)}(s)dW_s \right] \right], \quad (49) \]

which is the Radon-Nikodym density of the risk neutral probability \( Q^{(\lambda)} \) w.r.t. the historical probability \( \mathbb{P} \).

Let \( D^{(\lambda)} \) be the discount factor associated to \( \lambda \):

\[ D^{(\lambda)}(t) = \exp \left[ -\int_0^t (R(s) + \delta(\lambda_s)) ds \right] = \exp \left[ -\int_0^t R(s) ds \right] = D(t), \quad (50) \]

since \( \delta(\lambda_s) = 0 \).

Then, \( \kappa^{(\lambda)} \) is defined by:

\[ \kappa^{(\lambda)}(t) = D^{(\lambda)}(t)\eta^{(\lambda)}(t). \]
3.1.2 CRRA utility case

Suppose that \( U(x) = \frac{x^{1-\gamma}}{1-\gamma} \). Then, we have for any \( \nu \in \mathcal{D} \):

\[
V^{(\nu)}(T) = V_0 \frac{[I_T\kappa^{(\nu)}(T)]^{(1-\frac{1}{\gamma})}}{\mathbb{E}_P [(I_T\kappa^{(\nu)}(T))^{(1-\frac{1}{\gamma})}]}.
\]

Additionally, the Legendre-Fenchel transformation \( \hat{U} \) of the utility function \( U \) is given by:

\[
\hat{U}(y) = U(J(y) - yJ(y)) = y^{(1-\frac{1}{\gamma})} \frac{\gamma}{1-\gamma}.
\]  

(51)

We also have:

\[
F^{(\nu)}(y) = \mathbb{E}_P \left[ J \left[ yI_T\kappa^{(\nu)}(T) \right] \kappa^{(\nu)}(T) \right] = y^{(-\frac{1}{\gamma})} \mathbb{E}_P \left[ (I_T\kappa^{(\nu)}(T))^{(1-\frac{1}{\gamma})} \right].
\]

Thus :

\[
F^{(\nu)-1}(V_0) = \left( \frac{V_0}{\mathbb{E}_P [(I_T\kappa^{(\nu)}(T))^{(1-\frac{1}{\gamma})}]} \right)^{-\gamma}.
\]

Therefore, we get :

\[
\mathbb{E}_P \left[ \kappa^{(\nu)}(T) \hat{U} \left( F^{(\nu)-1}(V_0)\kappa^{(\nu)}(T) \right) \right] = \frac{\gamma}{1-\gamma} V_0^{1-\gamma} \left( \mathbb{E}_P \left[ \kappa^{(\nu)}(T)^{(1-\frac{1}{\gamma})} \right] \right)^\gamma.
\]

As we take \( \gamma > 1 \), the minimization w.r.t. \( \nu \) of \( \mathbb{E}_P \left[ \kappa^{(\nu)}(T) \hat{U} \left( F^{(\nu)-1}(V_0)\kappa^{(\nu)}(T) \right) \right] \) is equivalent to the maximization of

\[
\left( \mathbb{E}_P \left[ \kappa^{(\nu)}(T)^{(1-\frac{1}{\gamma})} \right] \right)^\gamma.
\]

Note that, since \( \nu_s \) belongs to \( \tilde{K} \), we have: \( \delta (\nu_s) = 0 \).

But:

\[
\kappa^{(\nu)}(T) = \exp \left[ -\int_0^t R(s) ds - \frac{1}{2} \int_0^t \sigma(s) \sigma(s) dW, W \right] - \int_0^t N^{(\nu)}(s) dW_s,
\]

with

\[
N^{(\nu)}(s) = \left[ s + \Gamma^{-1}_f(s) \nu(s) \right] \Sigma_e \left[ N(s) + \Gamma^{-1}_f(s) \nu(s) \right].
\]

Thus :

\[
\kappa^{(\nu)}(T)^z = \exp \left[ -z - \int_0^t R(s) ds - \frac{1}{2} z \int_0^t N^{(\nu)}(s) \Sigma_e N^{(\nu)}(s) s dW_s - z \int_0^t N^{(\nu)}(s) dW_s \right].
\]
Additionally, the process $I$ is defined by:

$$I_t = I_0 \exp \left[ \int_0^t \sigma_t^2 s - \frac{1}{2} \sigma_t^2 t + \int_0^t W_s^t ds \right]$$

Thus we have:

$$\left[ I_{tK^{(\nu)}(t)} \right]^2 = \exp \left[ -z \int_0^t \left[ r(s) + \alpha'(0) \right] ds - \frac{1}{2} \zeta t \int_0^t N^{(\nu)}(s) \Sigma_c \Sigma^{(\nu)}(s) ds \right]$$

$$\times \exp \left[ -z \int_0^t N^{(\nu)}(s) dW_s - \frac{1}{2} \zeta t + z \sigma_1 \int_0^t W_s^t ds \right]$$

Note that this latter process is Gaussian.

Introduce the function $G$ defined by:

$$G(z, \nu, t) = \mathbb{E}_p \left[ \left( I_{tK^{(\nu)}(t)} \right)^2 \right] .$$

Due to the normality of the logarithm of $\left[ I_{tK^{(\nu)}(T)} \right]^2$, $G$ is a function the logarithm of which is quadratic w.r.t. $\nu$.

Indeed, we have:

$$\left[ I_{tK^{(\nu)}(t)} \right]^2 = \exp \left[ -\frac{1}{2} \zeta^2 \int_0^t N^{(\nu)}(s) \Sigma_c \Sigma^{(\nu)}(s) ds - z \int_0^t \sigma_t^2 s \right]$$

$$\times \exp \left[ z \int_0^t \left[ r(s) + \alpha'(0) \right] ds + \frac{1}{2} \left( z^2 - z \right) \int_0^t \left[ t N^{(\nu)}(s) \Sigma_c \Sigma^{(\nu)}(s) + \sigma_t^2 \right] ds \right]$$

where the first process is an exponential martingale with expectation equal to 1.

Therefore, if $\nu$ is deterministic:

$$\ln \left[ G(z, \nu, t) \right] = z \int_0^t \left[ r(s) + \alpha'(0) \right] ds + \frac{1}{2} \left( z^2 - z \right) \int_0^t \left[ t N^{(\nu)}(s) \Sigma_c \Sigma^{(\nu)}(s) + \sigma_t^2 \right] ds + z \sigma_1 \int_0^t N^{(\nu)}(s) d\langle W, W^t \rangle_s ,$$

which indeed is quadratic w.r.t. $\nu$.

The optimization problem is equivalent to the maximization of the function:

$$P(\nu, t) = G \left( \left[ 1 - \frac{1}{\gamma} \right], \nu, t \right)^\gamma ,$$
for any \( v \in \tilde{K} \), and for any \( t \in [0, T] \).

Thus, the process \( \lambda \) is determined from:

\[
\lambda(t) = \arg \min_{v \in \tilde{K}} [P(v, t)],
\]

(54)

Usually, the determination of process \( \lambda \) is not easy. But if process \( \lambda(t) \) is deterministic, it is much easier (see Appendix):

\[
\lambda(t) = \begin{bmatrix} 0 \\ l(t) \\ 0 \\ 0 \end{bmatrix}
\]

with

\[
l(t) = - \frac{t}{\Sigma_{c}} \beta_{c} + (\gamma - 1) \left[ \frac{(0, 1, 0, 0). \left( t \Sigma \right)^{-1} t(-a_{r} \beta_{c}(T - t), 0, \sigma_{I}, 0)}{\Sigma_{c}} \right].
\]

For \( \gamma = 1 \), we recover the logarithmic value of \( l \).

Recall that \( P = \Gamma^{-1} t(0, 1, 0, 0) \). Set: \( P = t(p_{r}, p_{I}, p_{S}) \).

The dynamic programming approach (see next section) allows us to claim that, when looking at optimal weights, \( \lambda(t) \) is indeed deterministic. Therefore, we get:

**Proposition 6** For the CRRA case, and within the framework of our model, the optimal portfolio value, without investment on the indexed-inflation bond, is given by:

\[
V^{(\lambda)}(T) = V_{0} \frac{[I_{TK}(\lambda)(T)](\frac{1}{1 - \gamma})}{\mathbb{E}_{P} [I_{TK}(\lambda)(T)](1 - \frac{1}{\gamma})}.
\]

(55)

Optimal weights are given by (see Appendix):

\[
\begin{bmatrix} x^{CRRA}_{B}(t) \\ 0 \\ x^{CRRA}_{C}(t) \\ x^{CRRA}_{S}(t) \end{bmatrix} = \frac{1}{\gamma} \left[ t \Sigma \right]^{-1} \begin{bmatrix} \lambda_{r} + l(t)p_{r} \\ \lambda_{i} + l(t)p_{I} \\ \lambda_{I} + l(t)p_{I} \\ \lambda_{S} + l(t)p_{S} \end{bmatrix} + (1 - \frac{1}{\gamma}) \left[ t \Sigma \right]^{-1} \begin{bmatrix} -a_{r} \beta_{c}(T - t) \\ 0 \\ \sigma_{I} \\ 0 \end{bmatrix},
\]

where \( l(t) \) is equal to:

\[
l(t) = - \frac{t}{\Sigma_{c}} \beta_{c} + (\gamma - 1) \left( (0, 1, 0, 0). \left( t \Sigma \right)^{-1} t(-a_{r} \beta_{c}(T - t), 0, \sigma_{I}, 0) \right).\]
3.2 Optimal weight (HJB approach)

To solve the optimization problem given $V_0$, where the set of variables is the collection of all self-financing strategies, we use the method of dynamic programming, as proposed by Merton (1971).

Recall that the portfolio value is solution of the following (SDE):

$$
\frac{dV_t}{V_t} = \left[ (R_t - i_t + \sigma^2_t) + \bar{\xi}(t), \tilde{\vartheta}_t - \sigma_1 \bar{\Gamma}_t \right] dt + \frac{t}{\Sigma_t} \sim dW_t - \sigma_1 dW_t^I,
$$

where $t$ and $\tilde{D}_t = (0, 0, 1, 0)$.

Therefore, it is solution of

$$
dV_t = a(t, V_t, \bar{\xi}(t)) dt + b(t, V_t, \bar{\xi}(t)) dW_t,
$$

with:

$$
a(t, V_t, \bar{\xi}(t)) = V_t \left[ (R_t - i_t + \sigma^2_t) + \bar{\xi}(t), \tilde{\vartheta}_t - \sigma_1 \bar{\Gamma}_t \right],
$$

and

$$
b(t, V_t, \bar{\xi}(t)) = V_t \left[ \bar{\xi}(t) - \tilde{D}_t \right] \Sigma_t.
$$

Denote $F_t = (r_t, i_t)$ the vector of basic financial factors. We have:

$$
dF_t = A_t dt + B_t dW_t^F,
$$

with $W_t^F = t(W_t^R, W_t^I)$ and

$$
B_t = \begin{bmatrix} a_r & 0 \\ 0 & a_i \end{bmatrix} \text{ and } A_t = \begin{bmatrix} k_r (\tau - r_t) \\ k_i (\tau - i_t) \end{bmatrix}.
$$

Let $J(t, T, V_t, F_t)$ denote the value function over the time period $[t, T]$ with wealth value $V_t$ at time $t$ and given the factor value $F_t$ at time $t$. We have:

$$
J(t, T, V_t, F_t) = \max_{x \in \mathbb{R}, s \in C_{[t, T]}^1} \mathbb{E} [U(V_T)]. \quad (56)
$$

The strategy is assumed to be such that $\bar{x}_t = \bar{x}(t, V_t, F_t)$ where $\bar{x}(., ., .)$ is a deterministic function on $[0, T] \times \mathbb{R} \times \mathbb{R}$.

Denote:

$$
f(t, v, x) = (a(t, v, x), A_t) \text{ and } g(t, v, x) = (b(t, v, x), B_t).
$$

The Hamiltonian $H(., ., .)$ associated to Problem (56) is given by:\footnote{See Merton (1990), Prigent (2007) for more details about dynamic programming method based on Hamilton-Jacobi-Bellman equation.}

$$
H(t, v, p, q, x) = \mathbb{E} [U(t, v, x) + \langle p, f(t, v, x) \rangle + \frac{1}{2} \text{Trace} \left( g(t, v, x)^T q g(t, v, x) \right)].
$$
where:

\[ p = \left( \frac{\partial J}{\partial v}(t,v,F), \frac{\partial J}{\partial F}(t,v,F) \right), \]

and

\[ q = \begin{bmatrix} \frac{\partial^2 J}{\partial v^2}(t,v,F) & \frac{\partial^2 J}{\partial v \partial F}(t,v,F) \end{bmatrix}. \]

Applying the dynamic programming approach, the function value must satisfy the Hamilton-Jacobi-Bellman (HJB) condition:

\[ 0 = \frac{\partial J}{\partial t}(t,v) + H^* \left( t,v, \frac{\partial J}{\partial v}(t,v), \frac{\partial^2 J}{\partial v^2}(t,v) \right), \]

jointly with

\[ J(T,T,V_T,F_T) = U(V_T), \]

where

\[ H^* \left( t,v, \frac{\partial J}{\partial v}(t,v), \frac{\partial^2 J}{\partial v^2}(t,v) \right) = \max_{x} H \left( t,v, \frac{\partial J}{\partial v}(t,v), \frac{\partial^2 J}{\partial v^2}(t,v), x \right). \]

Here, due to stochastic factors, this is equivalent to:

\[ 0 = \frac{\partial J}{\partial t}(t,v) + \max_{x} H \left( t,v, \frac{\partial J}{\partial v}(t,v), \frac{\partial^2 J}{\partial v^2}(t,v), x \right) \]

\[ + \frac{1}{2} \left[ \left( R_t - i_t + \sigma_t^2 \right) \mathbf{x}(t), \left( \tilde{\sigma}_t - \sigma_t \tilde{\Gamma}_t, D_t^F \right) \right] \frac{\partial J}{\partial v}(t,v) V_t^2 \]

\[ + \frac{1}{2} \left[ \left( \mathbf{x}(t) - \tilde{D}_t \right), \tilde{S}_t, \tilde{S}_t, \left( \mathbf{x}(t) - \tilde{D}_t \right) \right] \frac{\partial^2 J}{\partial v^2}(t,v) V_t^2 \]

\[ + \frac{1}{2} \tilde{A}_t \frac{\partial J}{\partial F}(t,v) + \frac{1}{2} \sum_{i,j=1}^2 b_{i,j} D^F \tilde{F}_t \frac{\partial^2 J}{\partial F^2}(t,v), \quad (57) \]

where

\[ D^F dt = d(W^F)_t, D^{I,F} dt = d(W^I, W^F)_t, \]

and

\[ b_{i,j} \text{ denotes the } i \text{-th row of the matrix } B_t. \]

To determine the optimal portfolio for the CRRA case, note that it is independent from the initial wealth \( V_t \) since the CRRA utility function is homothetic and the processes \( \frac{dV}{V} \) and \( dF \) do not depend on \( V \). We have:

\[ J(t,T,V_t,F_t) = V_t^{(1-\gamma)} \max_{x \in [t,T]} \mathbb{E} \left[ U \left( \frac{V_T}{V_t} \right) | \mathcal{F}_t \right] = V_t^{(1-\gamma)} J(t,T,1,F_t). \]

Thus, we get:

\[ J(t,T,V_t,F_t) = U(V_t) \Theta(t,T,F_t)^\gamma, \]

with

\[ \Theta(t,T,F_t)^\gamma = (1-\gamma) J(t,T,1,F_t). \]

(59)

(60)
Now, we can apply the first order condition to $\bar{x}(t)$ in Equation (57). We get:

$$\bar{x}(t)^* =$$

$$\bar{D}^t + \left( \Sigma_t \Sigma_c \cdot \Sigma_t \right)^{-1} \left( -\frac{\partial J}{\partial t} (t,v) V_t \left( \theta_t - \sigma_I \theta_{\bar{t}}, D^t \right) - \frac{\partial^2 J}{\partial v^2} (t,v) V_t^2 \left( \Sigma_t \Sigma F^t B \right) \right).$$

Denote by $K_t$ the vector: $^t K_t = (-a_r \beta, \sigma_I, \sigma_S \rho^I, S)$. We deduce:

$$\bar{x}(t)^* = \left( \Sigma_t \Sigma_c \cdot \Sigma_t \right)^{-1} \left( -\frac{\partial J}{\partial t} (t,v) V_t \left( \theta_t \right) - \frac{\partial^2 J}{\partial v^2} (t,v) V_t^2 \left[ \Sigma_t \Sigma F^t B \right] + \frac{\partial J}{\partial F} (t,v) V_t + \frac{\partial^2 J}{\partial v^2} (t,v) V_t^2 \sigma_I \theta_{\bar{t}}, D^t \right).$$

Then, we have:

$$\bar{x}(t)^* = \left( \Sigma_t \Sigma_c \cdot \Sigma_t \right)^{-1} \left( \frac{1}{\gamma} \left( \theta_t \right) + \Sigma_t \Sigma F^t B \cdot \left[ \frac{1}{\partial F} \frac{\partial \theta}{\partial F} \right] - \left( \frac{1 - \gamma}{\gamma} \right) \sigma_I K_t \right). \quad (61)$$

Thus, the HJB equation is as follows:

$$0 = \frac{\partial \theta}{\partial t} + ^t A_t \frac{\partial \theta}{\partial F}$$

13Note that, from previous relation between $J$ and $\theta$, we have:

$$\frac{\partial J}{\partial t} = \gamma \frac{\partial \theta}{\partial t} \frac{1}{\theta} J,$$

$$\frac{\partial J}{\partial v} = (1 - \gamma) J,$$

$$\frac{\partial^2 J}{\partial v^2} = (1 - \gamma)(-\gamma) J,$$

$$\frac{\partial J}{\partial F} = \gamma \frac{\partial \theta}{\partial F} \frac{1}{\theta} J,$$

$$\frac{\partial^2 J}{\partial v \partial F} = (1 - \gamma) \gamma \frac{\partial \theta}{\partial F} \frac{1}{\theta} J.$$
Since $W_t^F$ is a component of the vector $W_t$, then we have:

$$D^F \Sigma_e^{-1} D^F = D^{F,F}.$$  

Therefore, the non linear term in Equation (62) is equal to zero. Thus, The HJB equation is reduced to a linear second order PDE, as mentioned by Chiarella et al. (2007). Using Feynman-Kac formula, we determine function $\Phi$:

$$\Theta(t,T,r_t,i_t) = \exp \left[ \frac{1 - \gamma}{\gamma} B_r(T-t) r_t \right] \Delta(t,T),$$  

(63)

with

$$\Delta(t,T) = \exp \left[ \delta(T-t) + \left( \frac{1 - \gamma}{\gamma} \right) (T-t-B_r(T-t)) \left( \tau + \xi \frac{\sigma^I r_t}{k_r} \right) \right].$$  

with

$$\delta = \left( \frac{1 - \gamma}{2 \gamma^2} \right)^I \lambda \Sigma_e^{-1} \lambda + \left( \frac{1 - \gamma}{2 \gamma^2} \right) (\sigma^I)^2 - \left( \frac{1 - \gamma}{\gamma^2} \right)^I \lambda^I \sigma^I,$$

$$\xi = \left( \frac{1 - \gamma}{\gamma} \right) \left[ \lambda r - \sigma^I p_{r,t} \right].$$

Therefore, we get:

$$\frac{1}{\Phi} \partial \Theta \partial r = \left( \frac{1 - \gamma}{\gamma} \right) \frac{B_r(T-t)}{(T-t)} \text{ and } \frac{1}{\Phi} \partial \Theta \partial i = 0.$$  

Thus:

$$\frac{1}{\Phi} \partial \Theta \partial F = \begin{bmatrix} \left( \frac{1 - \gamma}{\gamma} \right) \frac{B_r(T-t)}{(T-t)} & 0 \\ 0 & 0 \end{bmatrix}.$$  

Then, substituting the expression $\Theta$ given in (63) into Equation (61), we deduce the solution.
Proposition 7  The optimal weights without the investment opportunity in the indexed-inflation bond are given by: $x_{B_i} = 0$ and

$$
\bar{x}(t)^* = \frac{1}{\gamma} \left( \Sigma_{\ell} \Sigma_{\epsilon} \Sigma_{\ell} \right)^{-1} \begin{pmatrix} \theta_B \\ \theta_C \\ \theta_S \end{pmatrix} \\
+ \left( 1 - \frac{1}{\gamma} \right) \left( \Sigma_{\ell} \Sigma_{\epsilon} \Sigma_{\ell} \right)^{-1} \begin{pmatrix} \rho_{r,I}^\prime \Sigma_{\ell} + \Sigma_{\ell} \Sigma_{\epsilon} \Sigma_{\ell} \Sigma_{\epsilon} \Sigma_{\ell} \end{pmatrix} \begin{pmatrix} \rho_{r,I}^\prime \\ \sigma_I \\ \sigma_S \end{pmatrix}.
$$

Remark 8  Therefore, the optimal portfolio can be expressed as follows:\(^{14}\)

$$
\bar{x}(t)^* = \frac{1}{\gamma} \bar{x}(1)(t) + \left( 1 - \frac{1}{\gamma} \right) \bar{x}(2)(t) + \left( 1 - \frac{1}{\gamma} \right) \bar{x}(3)(t),
$$

(64)

where $\bar{x}(1)$ corresponds to the Markowitz portfolio, $\bar{x}(2)$ to the intertemporal portfolio and finally $\bar{x}(3)$ to the inflation hedging portfolio. Note also that we have:

$$
\bar{x}(t)^* = \frac{1}{\gamma} \bar{x}(1) + \left( 1 - \frac{1}{\gamma} \right) \bar{x}(4),
$$

(65)

where $\bar{x}(4) = \bar{x}(2) + \bar{x}(3)$ denotes the conservative portfolio.

Recall that the optimal portfolio weight $x(t)^*$, when we can invest in the indexed-inflation bond, is given by:\(^{15}\)

$$
\begin{bmatrix} x_{CRRA}^{\ell}(t) \\ x_{CRRA}^{\epsilon}(t) \\ x_{CRRA}^{\ell}(t) \\ x_{CRRA}^{S}(t) \end{bmatrix} = \frac{1}{\gamma} \mu^{-1} \begin{pmatrix} \lambda_r \\ \lambda_i \\ \lambda_I \end{pmatrix} + \left( 1 - \frac{1}{\gamma} \right) \begin{pmatrix} -a_r \beta_r (T - t) \\ 0 \\ \sigma_I \end{pmatrix}.
$$

Thus, when we cannot invest in the indexed-inflation bond, the risk of the nominal price index is only hedged by its correlations with the other risky assets (see $\bar{x}(3)$). The investor is more exposed to the inflation risk, since no asset can provide a certain payoff. It can only partially hedge her financial position.

\(^{14}\)See Chiarella et al. (2007) for such kind of decomposition.

\(^{15}\)In that case, we deal with the matrix $\Sigma$ instead of $\Sigma$, and $\Sigma$ is invertible.
4 Compensating variation

In this section, we use the quantitative index of investor’s satisfaction introduced by de Palma and Prigent (2008, 2009) to measure the utility loss when the indexed-inflation bond is not available on the market. It is based on the standard economic concept of *compensating variation*.

If an investor with risk aversion \( \gamma \) and initial investment \( V_0 \) faces a market with the indexed-inflation bond, her expected utility is \( \mathbb{E}[U_\gamma(V^*_T); V_0] \). If this investor selects an optimal portfolio without the indexed-inflation bond, she will get the expected utility \( \mathbb{E}[U_\gamma(V^\gamma_T); V_0] \). She will get the same expected utility provided that she invests an initial amount \( \tilde{V}_0 \geq V_0 \). Therefore, this investor requires a compensation \( \tilde{V}_0 \) which satisfies:

\[
\mathbb{E}[U_\gamma(V^*_T); V_0] = \mathbb{E}[U_\gamma(V^\gamma_T); \tilde{V}_0].
\]

The amount \( \tilde{V}_0 \) is in line with the certainty equivalent concept in expected utility analysis. It can be viewed as an “implied initial investment” necessary to maintain the level of expected utility.

4.1 Logarithmic case

This equation can be solved explicitly and leads to:

\[
\frac{\tilde{V}_0}{V_0} = \exp \left[ \mathbb{E} \left[ \ln \kappa(\lambda)(T) \right] - \mathbb{E} \left[ \ln \kappa(T) \right] \right].
\]

(66)

Using definitions of both \( \kappa(\lambda)(T) \) and \( \kappa(T) \) and properties of the nominal numeraire portfolio, we get the following result.

**Proposition 9** For the logarithmic case, the compensating variation due to the lack of indexed-inflation bond is equal to:

\[
\frac{\tilde{V}_0}{V_0} = \exp \left[ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \Gamma_f^{-1} \lambda(t) \right) \Sigma_c \Gamma_f^{-1} \left[ -\lambda(t) - 2\theta(t) \right] dt \right] \right].
\]

(67)

**Proof.** Recall that

\[
\kappa(T) = \mathcal{E} \left[ -\int_0^T N(t) dW_t \right] \exp \left[ -\int_0^T R(t) dt \right],
\]

\[
\kappa(\lambda)(T) = \mathcal{E} \left[ -\int_0^T N(\lambda)(t) dW_t \right] \exp \left[ -\int_0^T R(t) dt \right],
\]

and \( N(\lambda)(t) = N(t) + \Gamma_f^{-1} \lambda(t) \), with \( N(t) = \Gamma_f^{-1} \theta(t) \).

Using Yor’s formula,16 we have:

\[
\mathcal{E} \left[ -\int_0^T N(\lambda)(t) dW_t \right] = \mathcal{E} \left[ -\int_0^T N(t) dW_t - \int_0^T \Gamma_f^{-1} \lambda(t) dW_t \right] =
\]

\[
\mathcal{E} \left[ -\int_0^T N(t) dW_t \right] - \mathcal{E} \left[ \int_0^T \Gamma_f^{-1} \lambda(t) dW_t \right].
\]

16Recall the Yor’s formula. For any two semimartingales \( X \) and \( Y \), we have:

\[
\mathcal{E}[X] \mathcal{E}[Y] = \mathcal{E}[X + Y + [X, Y]],
\]

where \([X, Y]\) denotes the quadratic variation of the two processes.
\[ \mathcal{E}[\int_0^T N(t)dW_t] \mathcal{E}[\int_0^T \Gamma_f^{-1} \lambda(t)dW_t] \exp[-\int_0^T t (\Gamma_f^{-1} \lambda(t)) N(t)d(W, W)_t]. \]

Additionally, by properties of the nominal numeraire portfolio (see Appendix B), we get:

\[ \mathbb{E} \left[ \ln \kappa^{(\lambda)}(T) \right] = \mathbb{E} \left[ \ln \kappa(T) \right] - \frac{1}{2} \mathbb{E} \left[ \int_0^T \left[ t (\Gamma_f^{-1} \lambda(t)) \Sigma^{-1} \lambda(t) + 2t (\Gamma_f^{-1} \lambda(t)) N(t) \right] d(W, W)_t \right]. \]  

(68)

Finally, using the correlation matrix \( \Sigma_c \) of the brownian motion \( W \), we deduce that the compensating variation is equal to:

\[ \frac{\tilde{V}_0}{V_0} = \exp \left[ \frac{1}{2} \mathbb{E} \left[ \int_0^T t (\Gamma_f^{-1} \lambda(t)) \Sigma_c \Gamma_f^{-1} [-\lambda(t) - 2\theta(t)] dt \right] \right]. \]  

(69)

Numerical illustration:

<table>
<thead>
<tr>
<th>Table 1: Compensating variation per year.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T = 10 )</td>
</tr>
<tr>
<td>CV</td>
</tr>
</tbody>
</table>

Table (1) shows how the compensating variation (per year) varies according to investment horizon \( T \) for the logarithmic case. We note that this compensation is decreasing with horizon. These "theoretical" costs can be compared to management costs approximately equal to 3% per year, which illustrates the impact of the lack of indexed bonds to hedge against inflation.

### 4.2 CRRA case

**Proposition 10** For the CRRA case, the compensating variation due to the lack of indexed-inflation bond is equal to:

\[ \frac{\tilde{V}_0}{V_0} = \frac{\mathbb{E}_P \left[ (I_T K(T))^{(1-\frac{1}{\gamma})} \right] (\frac{1}{\gamma})}{\mathbb{E}_P \left[ (I_T K(T))^{(1-\frac{1}{\gamma})} \right] (\frac{1}{\gamma})} = \left( \frac{G((1 - \frac{1}{\gamma}), 0, T)}{G((1 - \frac{1}{\gamma}), \lambda, T)} \right) (\frac{1}{\gamma}). \]  

(70)

**Remark 11** The compensating variation can be also expressed with the cash equivalent certain that we denote by \( c \). The cash \( c \) is defined from the relation:

\[ U(c) = \mathbb{E} [U(V_T)]. \]

Denote by \( \tilde{c} \) the cash equivalent certain corresponding to the optimal portfolio without the indexed-inflation bond. Thus, for both the logarithmic and CRRA cases, the compensating variation is the ratio of cash equivalents:

\[ \frac{\tilde{V}_0}{V_0} = \frac{c}{\tilde{c}}. \]
5 Conclusion

In this paper, we have studied the portfolio optimization problem in the presence of inflation risk when no perfect hedge against inflation exists. In this framework, we have provided the solution by using martingale and convexity properties, as introduced in Cvitanic and Karatzas (1992). We have also computed optimal weights by using the dynamic programming approach, based on the Hamilton-Jacobi-Bellmann equation. We have also provided a measure to quantify the loss due to the lack of inflation indexed bonds. This measure, which corresponds to the compensating variation, allows to illustrate the need for such bonds.

References


6 Appendix

6.1 Appendix A: the multifactor model

Condition (1): \( \left( \left( B(r_i, i_t, t, T) / \exp \left[ \int_0^t R_s ds \right] \right) \times \eta_t \right)_t \) is a \( \mathbb{P} \)-martingale:

\[
\mu(t, T - t) - R_t + \alpha_i \beta_i(T - t) \left[ \lambda_i + \lambda_r \rho^r, i + \rho^i, s \lambda_s \right] \\
+ \alpha_r \beta_r(T - t) \left[ \lambda_r + \lambda_i \rho^r, i + \lambda_r \rho^r, I + \rho^r, s \lambda_s \right] = 0.
\]

Condition (2): \( \left( \left( B^{I(n)}(r_i, i_t, t, T) / \exp \left[ \int_0^t R_s ds \right] \right) \times \eta_t \right)_t \) is a \( \mathbb{P} \)-martingale:

\[
\mu^{B^{I(n)}}(t, T - t) - R_t + \beta_{B^{I(n)}}(T - t) a_r \lambda_r - \sigma_I \lambda_I + \left[ -\sigma_I \lambda_r + \beta_{B^{I(n)}} a_r \lambda_I \right] \rho^{r, I} \\
+ \beta_{B^{I(n)}} a_r \lambda_r \rho^{r, i} + \left[ \beta_{B^{I(n)}} a_r \rho^{r, s} - \sigma_I \rho^{I, s} \right] \lambda_s = 0.
\]

Condition (3): \( \left( \left( C^{I(n)} / \exp \left[ \int_0^t R_s ds \right] \right) \times \eta_t \right)_t \) is a \( \mathbb{P} \)-martingale:

\[
r_t + i_t - R_t - \sigma_I \left[ \lambda_r \rho^{r, I} + \lambda_i \rho^{I, s} + \lambda_I \right] = 0.
\]

Condition (4): \( \left( \left( S(r_i, i_t, t, T) / \exp \left[ \int_0^t R_s ds \right] \right) \times \eta_t \right)_t \) is a \( \mathbb{P} \)-martingale:

\[
(\mu^S - R_t) - \sigma_s \left[ \lambda_r \rho^{r, s} + \lambda_i \rho^{I, s} + \lambda_I \right] = 0.
\]

From the previous system of equations, we deduce the risk premia of the four basic assets \( B, B^{I(n)}, C^{I(n)} \) and \( S \). They are defined as the excess instantaneous expected returns with respect to the nominal interest rate \( R \).

The four risk premia \( \theta^B, \theta^{B^{I(n)}}, \theta^{C^{I(n)}} \) and \( \theta^S \) are given by:

i) Nominal Bond \( B \):

\[
\theta^B(T - t) = \mu(t, T - t) - R_t = -\alpha_i \beta_i(T - t) \left[ \lambda_i + \rho^i, s \lambda_s \right] \\
- a_r \beta_r(T - t) \left[ \lambda_r + \lambda_i \rho^r, i + \lambda_r \rho^r, I + \rho^r, s \lambda_s \right].
\]

ii) Nominal value of the Indexed-Inflation Bond \( B^{I(n)} \):

\[
\theta^{B^{I(n)}}(T - t) = \mu^{B^{I(n)}}(t, T - t) - R_t = \\
\sigma_I \left[ \lambda_r \rho^{r, I} + \lambda_I + \rho^{I, s} \lambda_s \right] - a_r \beta_{B^{I(n)}}(T - t) \left[ \lambda_r + \lambda_i \rho^r, i + \lambda_r \rho^r, I + \rho^r, s \lambda_s \right].
\]

iii) Money account \( C^{I(n)} \):

\[
\theta^{C^{I(n)}} = r_t + i_t - R_t = \sigma_I \left[ \lambda_r \rho^{r, I} + \lambda_I + \lambda_s \rho^{I, s} \right]. \quad (71)
\]

iv) Stock price \( S \):
\[ \theta^S = \mu^S - R_t = \sigma_s [\lambda_r \rho^{r,s} + \lambda_i \rho^{i,s} + \lambda_l \rho^{l,s} + \lambda_s] . \]

where \( \mu(t, T - t) \) and \( \mu^{B(t)}(t, T - t) \) as defined in equation (17) and (18) and \( \lambda_r, \lambda_i, \lambda_l \) and \( \lambda_s \) are constant usually interpreted as the market prices of risk associated respectively with the sources of risk \( W_t^r, W_t^i, W_t^l \) and \( W_t^S \).

We determine now the five functions \( \alpha(.) \), \( \beta_r(.) \), \( \beta_i(.) \), \( \alpha_{B(t)}(.) \) and \( \beta_{B(t)}(.) \), which are involved in the two exponential affine models which describe both the nominal value of the bond and the real Indexed-Inflation bond.

The nominal value of the bond price satisfies:

\[
B(r_t, i_t, t, T) = \exp \left[ -\alpha(T - t) - \beta_r(T - t)r_t - \beta_i(T - t)i_t \right],
\]

with

\[
\beta_r(\tau) = \left( \frac{1 - \exp(-\tau k_r)}{k_r} \right) ,
\]

\[
\beta_i(\tau) = \left( \frac{1 - \exp(-\tau k_i)}{k_i} \right) .
\]

and

\[
\frac{\alpha(\tau)}{\tau} = a_r \left[ \frac{\tau}{a_r} - \frac{\lambda_r + \lambda_i \rho^{r,i} + \lambda_l \rho^{l,i} + \rho^{r,s} \lambda_s}{k_r} \right] \left[ 1 - \frac{1}{\tau k_r} + \frac{\exp(-\tau k_r)}{\tau k_r} \right] ,
\]

\[
- \sigma_I \left[ \lambda_r \rho^{r,I} + \lambda_l \rho^{l,s} \right] + a_i \left[ \frac{7}{a_i} - \frac{\lambda_i + \lambda_i \rho^{r,i} + \rho^{r,s} \lambda_s}{k_i} \right] \left[ 1 - \frac{1}{\tau k_i} + \frac{\exp(-\tau k_i)}{\tau k_i} \right] ,
\]

\[
- \frac{a_r^2}{2k_r^2} \left[ 1 - \frac{2 \left[ 1 - \exp(-\tau k_r) \right]}{\tau k_r} + \frac{1 - \exp(-2\tau k_r)}{2\tau k_r} \right] ,
\]

\[
- \frac{a_i^2}{2k_i^2} \left[ 1 - \frac{2 \left[ 1 - \exp(-\tau k_i) \right]}{\tau k_i} + \frac{1 - \exp(-2\tau k_i)}{2\tau k_i} \right] ,
\]

\[
\frac{a_r a_i \rho^{r,i}}{k_r k_i} \left[ 1 - \frac{1 - \exp(-\tau k_r)}{\tau k_r} \right] \left[ 1 - \frac{1 - \exp(-\tau k_i)}{\tau k_i} \right] + \frac{1 - \exp(-\tau (k_r + k_i))}{\tau (k_r + k_i)} \right] .
\]

The real Indexed-Inflation bond

\[
B^{I_r}(t, T) = \exp \left[ -\alpha_{B^{I_r}}(T - t) - \beta_{B^{I_r}}(T - t)r_t \right] ,
\]

with

\[
\beta_{B^{I_r}}(\tau) = \beta_r(\tau) = \left( \frac{1 - \exp(-\tau k_r)}{k_r} \right) ,
\]

and

\[
\frac{\alpha_{B^{I_r}}(\tau)}{\tau} = -\frac{a_r^2}{2k_r^2} \left[ 1 - \frac{2 \left[ 1 - \exp(-\tau k_r) \right]}{\tau k_r} + \frac{1 - \exp(-2\tau k_r)}{2\tau k_r} \right] ,
\]

\[
+ \left( 1 - \frac{1}{\tau k_r} + \frac{\exp(-\tau k_r)}{\tau k_r} \right) \left[ \frac{-\frac{a_r \rho^{r,I}}{k_r} \left( \alpha_r + \lambda_l \right) + \lambda_r + \lambda_i \rho^{r,i} + \rho^{r,s} \lambda_s}{\tau} \right] .
\]

In this setting, \( B(r_t, i_t, t, T) \) can be replicated using the assets \( C \), \( B_D \) and \( B^{D(t)_n} \) which span the bond market, as in BJP (2001) for a simpler case. This dynamic combination of fixed-income securities of different durations is referred to as the passive immunization (see Fong, 1990; Fabozzi, 1996).
6.2 Appendix B: Properties of the numeraire portfolio

Recall that the Radon-Nikodym density of the risk-neutral probability is given by:

\[ \eta_t = \exp \left[ -M_t - \Lambda t \right] \]

where

\[ M_t = \int_0^t \lambda_t dW^r + \int_0^t \lambda_i dW^i + \int_0^t \lambda_l dW^l + \int_0^t \lambda_S dW^S, \]

and

\[ \Lambda = \frac{1}{2} \left( \lambda_r^2 + \lambda_i^2 + \lambda_l^2 + \lambda_S^2 \right) + \lambda_r \lambda_i \rho^{r,i} + \lambda_r \lambda_l \rho^{r,l} + \lambda_r \lambda_S \rho^{r,S} + \lambda_i \lambda_S \rho^{i,S} + \lambda_l \lambda_S \rho^{l,S}. \]

The numeraire portfolio \( H \) is equal to:

\[ H_t = \exp \left( \int_0^t R_s \, ds \right) / \eta_t. \]

Consequently, we get:

\[ \ln H_t = \int_0^t R_s \, ds + M_t + \Lambda t. \]

Since \((W^r, W^i, W^l, W^S)\) is a Gaussian process, we can deduce that \( \ln H_t \) is Gaussian itself.

We have to determine its expectation and variance. More precisely, we determine the conditional expectations \( \mathbb{E}_t \left[ \frac{H_z^T}{H_z^T} \right] \), for powers \( z \).

The ratio \( \frac{H_z^T}{H_z^T} \) satisfies:

\[ \frac{H_z^T}{H_z^T} = \exp \left[ N_z (t, T) \right], \]

with

\[ N_z (t, T) = \int_t^T R_s \, ds + M_T - M_t + \Lambda (T - t). \]

The process \( N_z (t, T) \) is Gaussian distributed. Thus, to determine \( \mathbb{E}_t \left[ \frac{H_z^T}{H_z^T} \right] \), we have just to compute \( \mathbb{E}_t \left[ N_z (t, T) \right] \) and \( \text{Var}_t \left[ N_z (t, T) \right] \) since we have:

\[ \mathbb{E}_t \left[ \frac{H_z^T}{H_z^T} \right] = \exp \left[ \mathbb{E}_t \left[ N_z (t, T) \right] + \frac{1}{2} \text{Var}_t \left[ N_z (t, T) \right] \right]. \]

1) Determination of \( \mathbb{E}_t \left[ N_z (t, T) \right] \).

We have:

\[ \mathbb{E}_t \left[ N_z (t, T) \right] = z \left( \int_t^T \mathbb{E}_u \left[ R_u \right] \, du + \Lambda (T - t) \right). \]

Recall that \( R_s = r_s + i_s + \alpha'(0) \). Then, we deduce:

\[ \mathbb{E}_t \left[ R_u \right] = \mathbb{E}_t \left[ r_u \right] + \mathbb{E}_t \left[ i_u \right] + \alpha'(0) = r + (r_t - r)e^{-(u-t)\kappa_r} + \tilde{i} + (i_t - \tilde{i})e^{-(u-t)\kappa_i} + \alpha'(0). \]

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Finally, we get:

$$\mathbb{E}_t [N_z(t, T)] = z(T-t) \left( \mathbf{r} + (r_t - \mathbf{r}) \frac{\beta_s(T-t)}{T-t} + i + (i_t - \mathbf{i}) \frac{\beta_s(T-t)}{T-t} + \alpha'(0) + \Lambda \right),$$

which is equivalent to:

$$\mathbb{E}_t [N_z(t, T)] = z \Phi(t, T),$$

with:

$$\Phi(t, T) = (T-t) \left[ \mathbf{r} + (r_t - \mathbf{r}) \frac{\beta_s(T-t)}{T-t} + i + (i_t - \mathbf{i}) \frac{\beta_s(T-t)}{T-t} + \alpha'(0) + \Lambda \right],$$

and

$$\alpha'(0) = -\sigma I [\lambda_r \rho^{r,t} + \lambda_I + \lambda_s \rho^{I,s}].$$

2) Determination of $\text{Var}_t [N_z(t, T)]$.

Step 1: computation of $\text{Var}_t \left[ \int_t^T R_s \, ds \right]$.

We have:

$$\text{Var}_t \left[ \int_t^T R_s \, ds \right] = \text{Var}_t \left[ \int_t^T r_s \, ds \right] + \text{Var}_t \left[ \int_t^T i_s \, ds \right]$$

$$+2 \text{Cov} \left[ \int_t^T r_s \, ds; \int_t^T i_s \, ds \right].$$

- To calculate $\text{Var}_t \left[ \int_t^T r_s \, ds \right]$, we use the Fubini's property for stochastic integral. We have:

$$r_t = \mathbf{r} + (r_s - \mathbf{r}) e^{-(s-t)k_r} + a_r e^{-(s-t)k_r} \int_s^t e^{(u-t)k_r} dW_v^r.$$

$$\int_t^T r_u \, du = \mathbf{r}(T-t) + (r_t - \mathbf{r}) \beta_r(T-t)$$

$$+ \int_t^T \left( a_r e^{-(u-t)k_r} \int_u^t e^{(v-t)k_r} dW_v^r \right) du.$$

Then:

$$\int_t^T \left[ a_r e^{-(u-t)k_r} \int_u^t e^{(v-t)k_r} dW_v^r \right] du = \int_t^T \left( \int_u^T a_r e^{-(u-t)k_r} e^{(v-t)k_r} du \right) dW_v^r,$$

$$= \int_t^T \left( a_r \left[ 1 - e^{-(T-t)k_r} \right] \right) dW_v^r,$$

$$\text{Var}_t \left[ \int_t^T r_s \, ds \right] = a_r^2 \int_t^T \left[ \frac{1 - e^{-(T-t)k_r}}{k_r} \right]^2 dv,$$

$$\text{Var}_t \left[ \int_t^T r_s \, ds \right] = \frac{a_r^2}{k_r^2} \left( (T-t) + \frac{1}{2k_r} [1 - \exp(-2(T-t)k_r)] - 2\beta_r(T-t) \right).$$
Thus:

\[ \text{Step 3: computation of } \text{Var}_t \left[ \int_t^T i_s \, ds \right] : \]

\[
\text{Var}_t \left[ \int_t^T i_s \, ds \right] = \frac{a_i^2}{k_i^2} \left[ (T-t) + \frac{1 - \exp(-2(T-t)k_i)}{2k_i} \right] - 2\beta_i(T-t) .
\]

- Finally, we determine \( \text{Cov}_t \left[ \int_t^T r_s \, ds; \int_t^T i_s \, ds \right] : \)

\[
\text{Cov}_t \left[ \int_t^T r_s \, ds; \int_t^T i_s \, ds \right] = a_r a_i \int_t^T \left( \frac{1 - e^{-(T-v)k_r}}{k_r} \right) \left( \frac{1 - e^{-(T-v)k_i}}{k_i} \right) \rho^{r,i} \, dv
\]

\[
= \frac{a_r a_i}{k_r k_i} \rho^{r,i} \left( (T-t) + \beta_r(T-t) + \beta_i(T-t) + \frac{1 - \exp[-(T-t)(k_r + k_i)]}{k_r + k_i} \right) .
\]

\[ \text{Step 2: computation of } \text{Var}_t \left[ \mathbf{M}_T - \mathbf{M}_t \right] : \]

\[ \text{Var}_t \left[ \mathbf{M}_T - \mathbf{M}_t \right] = 2\Lambda(T-t) . \]

\[ \text{Step 3: computation of } \text{Cov}_t \left[ \int_t^T R_s \, ds; \mathbf{M}_T - \mathbf{M}_t \right] : \]

\[
\text{Cov}_t \left[ \int_t^T R_s \, ds; \mathbf{M}_T - \mathbf{M}_t \right] = a_r \left( \lambda_r + \lambda_i \rho^{r,i} + \lambda_I \rho^{r,I} + \lambda_S \rho^{r,S} \right) \int_t^T \left[ 1 - \frac{e^{-(T-v)k_r}}{k_r} \right] \, dv
\]

\[
+ a_i \left( \lambda_r \rho^{r,i} + \lambda_i + \lambda_S \rho^{i,S} \right) \int_t^T \left[ 1 - \frac{e^{-(T-v)k_i}}{k_i} \right] \, dv,
\]

\[
= \frac{a_r}{k_r} \left( \lambda_r + \lambda_i \rho^{r,i} + \lambda_I \rho^{r,I} + \lambda_S \rho^{r,S} \right) \left( (T-t) + \beta_r(T-t) \right)
\]

\[
+ \frac{a_i}{k_i} \left( \lambda_r \rho^{r,i} + \lambda_i + \lambda_S \rho^{i,S} \right) \left( (T-t) + \beta_i(T-t) \right) .
\]

To conclude, recall that:

\[
\text{Var}_t \left[ N_z(t,T) \right] = z^2 \left( \text{Var}_t \left[ \int_t^T R_s \, ds \right] + \text{Var}_t \left[ \mathbf{M}_T - \mathbf{M}_t \right] + 2\text{Cov}_t \left[ \int_t^T R_s \, ds; \mathbf{M}_T - \mathbf{M}_t \right] \right) .
\]

Thus:

\[ \text{Var}_t \left[ N_z(t,T) \right] = z^2 \Psi_{t,T} , \]

with:

\[
\Psi_{t,T} = \frac{a^2}{k_i^2} \left[ (T-t) + \frac{1}{2k_i} \left[ 1 - \exp(-2(T-t)k_i) \right] - 2\beta_i(T-t) \right]
\]

\[
+ \frac{a^2}{k_r^2} \left[ (T-t) + \frac{1}{2k_r} \left[ 1 - \exp(-2(T-t)k_i) \right] - 2\beta_i(T-t) \right]
\]

\[
+ \frac{2a_r a_i}{k_r k_i} \rho^{r,i} \left( (T-t) + \beta_r(T-t) + \beta_i(T-t) + \frac{1 - \exp[-(T-t)(k_r + k_i)]}{k_r + k_i} \right)
\]

\[
+ 2\Lambda(T-t) + 2a_r \left( \lambda_r + \lambda_i \rho^{r,i} + \lambda_I \rho^{r,I} + \lambda_S \rho^{r,S} \right) \left( \frac{T-t + \beta_r(T-t)}{k_r} \right)
\]

\[
+ 2a_i \left( \lambda_r \rho^{r,i} + \lambda_i + \lambda_S \rho^{i,S} \right) \left( \frac{T-t + \beta_i(T-t)}{k_i} \right) .
\]
6.3 Appendix B': Properties of the real numeraire portfolio

The optimal portfolio value is given by:

$$V_T^{(n)*} = IJ \left( \lambda \frac{I_T}{H_T} \right) \text{ et } V_T^{(r)*} = J \left( \lambda \frac{I_T}{H_T} \right).$$

Therefore, we have to determine conditional expectations such as:

$$E_t \left[ \left( \frac{H_T}{I_T} \right)^z \right].$$

We have:

$$\frac{dI_t}{I_t} = r_t dt + \sigma_I dW_t^I,$$

then

$$I_t = I_0 \exp \left( \int_0^t r_s ds - \frac{1}{2} \sigma_I^2 I_t + \sigma_I W_t^I \right),$$

and:

$$\left( \frac{I_T}{I_t} \right)^{-z} = \exp \left[ -z \left( \int_t^T r_s ds - \frac{1}{2} \sigma_I^2 (T - t) + \sigma_I \int_t^T dW_t^I \right) \right] = \exp (-L_z(t, T)),$$

with

$$L_z(t, T) = z \left( \int_t^T r_s ds - \frac{1}{2} \sigma_I^2 (T - t) + \sigma_I \int_t^T dW_t^I \right).$$

Additionally (see previous Appendix B):

$$\left( \frac{H_T}{H_t} \right)^z = \exp \left[ N_z(t, T) \right], \text{ with } N_z(t, T) = z \left( \int_t^T R_s ds + M_T - M_t + \Lambda(T - t) \right).$$

Then we deduce:

$$\left( \frac{H_T}{H_t} \right)^z \left( \frac{I_T}{I_t} \right)^{-z} = \exp \left[ N_z(t, T) - L_z(t, T) \right].$$

The process $N_z(t, T) - L_z(t, T)$ is gaussian. Thus, our problem is just to compute the conditional expectation of a lognormal random variable.

1) Determination of the conditional expectation $E_t [N_z(t, T)] - E_t [L_z(t, T)]$:

We have:

$$E_t [N_z(t, T)] - E_t [L_z(t, T)] = z \Phi(t, T) - E_t [L_z(t, T)],$$

with:

$$E_t [i_u] = E[i_u | i_t] = t + (i_t - t) e^{-(u-t)k_t}.$$
\[
E_t [L_z(t, T)] = E_t \left[ z \left( \int_t^T i_u \, du - \frac{1}{2} \sigma_i^2 (T - t) + \sigma_i \int_t^T dW^i \right) \right] \\
= z \left( \int_t^T E_t[i_u] \, du - \frac{1}{2} \sigma_i^2 (T - t) \right) \\
= z \left( (T - t) \bar{i} + (i_t - \bar{i}) \int_t^T e^{-(T - u)k_i} \, du - \frac{1}{2} \sigma_i^2 (T - t) \right) \\
= z \left( (T - t) \bar{i} + (i_t - \bar{i}) \left[ \frac{e^{-(T - t)k_i}}{k_i} \right] T - \frac{1}{2} \sigma_i^2 (T - t) \right) \\
= z \left( (T - t) \bar{i} + (i_t - \bar{i}) \frac{1 - e^{-(T - t)k_i}}{k_i} - \frac{1}{2} \sigma_i^2 (T - t) \right) \\
= z \left( (T - t) \bar{i} + (i_t - \bar{i}) \beta_i (T - t) - \frac{1}{2} \sigma_i^2 (T - t) \right)
\]

We deduce that:

\[
E_t [N_z(t, T)] - E_t [L_z(t, T)] = \]

\[
= z \Phi(t, T) - z \left( (T - t) \bar{i} + (i_t - \bar{i}) \beta_i (T - t) - \frac{1}{2} \sigma_i^2 (T - t) \right), \\
= z \left( \int_t^T E_t [r_u] \, du + \int_t^T E_t [i_u] \, du + \int_t^T \alpha'(0) \, du + \Lambda(T - t) - \int_t^T E_t[i_u] \, du - \frac{1}{2} \sigma_i^2 (T - t) \right), \\
= z \left( \int_t^T E_t [r_u] \, du + \int_t^T \alpha'(0) \, du + \Lambda(T - t) + \frac{1}{2} \sigma_i^2 (T - t) \right) \bar{v} + (r_t - \bar{v}) \frac{\beta_r (T - t)}{(T - t)}. \\
\]

Finally we get:

\[
E_t [N_z(t, T)] - E_t [L_z(t, T)] = z \bar{\Phi}(t, T),
\]

with

\[
\bar{\Phi}(t, T) = (T - t) \left[ \bar{v} + (r_t - \bar{v}) \frac{\beta_r (T - t)}{(T - t)} + \alpha'(0) + \Lambda + \frac{1}{2} \sigma_i^2 \right].
\]

2) Determination of the conditional variance:

We use the standard decomposition:

\[
Var_t [N_z(t, T) - Var_t L_z(t, T)] = Var_t [N_z(t, T)] + Var_t [L_z(t, T)] - 2 Cov_t [N_z(t, T), L_z(t, T)].
\]

Note that:

\[
(1) \quad Var_t [N_z(t, T)] = z^2 \Psi(t, T).
\]

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\[ \text{Var}_t [L_z(t, T)] = z^2 Y(t, T) \text{ with } Y(t, T) = \left( \text{Var}_t \left[ \int_t^T i_s ds \right] + \sigma^2 I(T - t) \right). \]

(3)

\[ \text{Cov}_t [N_z(t, T), L_z(t, T)] = \text{E}_t [N_z(t, T) \times L_z(t, T)] - \text{E}_t [N_z(t, T)] \text{E}_t [L_z(t, T)], \]

and:

\[ z^2 \text{E}_t \left[ \left( \int_t^T R_s ds + M_T - M_t + \Lambda(T - t) \right) \left( \int_t^T i_s ds - \frac{1}{2} \sigma^2 I(T - t) + \sigma I \int_t^T dW^I \right) \right] = z^2 \times \]

\[ \text{E}_t \left[ \left( \int_t^T r_s ds + \int_t^T i_s ds + (M_T - M_t) + (\Lambda + \alpha'(0))(T - t) \right) \left( \int_t^T i_s ds - \frac{1}{2} \sigma^2 I(T - t) + \sigma I \int_t^T dW^I \right) \right]. \]

Therefore, we have:

\[ \text{E}_t [N_z(t, T) L_z(t, T)] = - \frac{z}{2} \sigma^2 I(T - t) \text{E}_t [N_z(t, T)] \]

\[ + z^2 \sigma I \left[ (\lambda_r \rho^{l,r} + \lambda_t + \lambda_s \rho^{l,s}) (T - t) + \int_t^T a_r \rho^{l,r} \beta_r(T - v)(v) dv \right] \]

\[ + z^2 (\Lambda + \alpha'(0))(T - t) \left( \int_t^T \text{E}_t [i_s] ds \right) \]

\[ + z^2 \int_t^T a_i \left[ \lambda_r \rho^{l,r} + \lambda_t + \lambda_s \rho^{l,s} \right] \beta_i(T - v)(v) dv \]

\[ + z^2 \text{E}_t \left[ \int_t^T r_s ds \int_t^T i_s ds \right] + z^2 \text{E}_t \left[ \left( \int_t^T i_s ds \right)^2 \right] \]

Finally we get:

\[ \text{Var}_t [N_z(t, T) - \text{Var}_t L_z(t, T)] = z^2 \bar{\Psi}(t, T), \]

with:

\[ \bar{\Psi}(t, T) = \frac{\alpha_r^2}{k_r^2} \left[ (T - t) + \frac{1}{2k_r} [1 - \exp(-2(T - t)k_r)] - 2 \beta_r(T - t) \right] \]

\[ + 2 \Lambda(T - t) + 2a_r (\lambda_r + \lambda_i \rho^{r,i} + \lambda_t \rho^{r,t} + \lambda_s \rho^{r,s}) \frac{(T - t) + \beta_r(T - t)}{k_r} \]

\[ + \sigma^2 I(T - t) - 2 \sigma_I a_r (\rho^{r,l}) \frac{(T - t) + \beta_r(T - t)}{k_r} - 2 \sigma_I (\lambda_r \rho^{r,l} + \lambda_t + \lambda_s \rho^{l,s}) \]

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To conclude, the $z$-power of the real numeraire portfolio value is given by:

$$\left( \frac{H_T}{H_t} \right)^z \left( \frac{I_T}{I_t} \right)^{-z} = \exp [N_z(t, T) - L_z(t, T)],$$

with:

$$\mathbb{E}_t \left[ \left( \frac{H_T}{H_t} \right)^z \left( \frac{I_T}{I_t} \right)^{-z} \right] = \exp \left[ z\Phi(t, T) + \frac{1}{2}z^2\Phi(t, T) \right].$$

### 6.3.1 Optimal CRRA weights (martingale method)

We determine $dV^{(v)CRRA}_t$ by using Relation (55). Applying Ito formula, we get the martingale part $dV^{(v)CRRA}_t$:

$$dV^{(v)CRRA}_t = (...)dt + \frac{1}{\gamma}V^{(v)CRRA}_t \left( \lambda_r dW^r + \lambda_i dW^i + (\lambda_I - \sigma_I) dW^I + \lambda_S dW^S + l(s) 'PdW \right)$$

$$+ V^{(v)CRRA}_t z [\beta_r a_r dW^r].$$

By identifying the martingale parts (here, the Brownian integrals), we get the following system:

$$\begin{cases}
\frac{\lambda_r}{\gamma} + za_r \beta_r (T-t) + l(s) \frac{p_r}{\gamma} = \delta_r(t) = -x_B(t) \beta_r(D)a_r - x_B(t) a_r \beta_B^{(v)}(D_i) \\
\frac{\lambda_i}{\gamma} + l(s) \frac{p_i}{\gamma} = \delta_i(t) = -x_B(t) a_i \beta_i(D) \\
\frac{\lambda_I - \sigma_I}{\gamma} + l(s) \frac{p_I}{\gamma} = \delta_I = x_C(t) \sigma_I + x_{B_i}(t) \sigma_I - \sigma_I \\
\frac{\lambda_S}{\gamma} + l(s) \frac{p_S}{\gamma} = \delta_S = x_S(t) \sigma_S
\end{cases}$$

where $P = t(p_r, p_i, p_I, p_S)$ and $x_{B_i}(t) = 0$, since the indexed bond is not available on the financial market.

Recall that $\Sigma$ denotes the matrix associated to the rates of return. Its transpose $^t\Sigma$ is given by:

$$^t\Sigma = \begin{bmatrix}
-\beta_r(D) a_r & -a_r \beta_B^{(v)}(D_i) & 0 & 0 \\
-\beta_i(D) a_i & 0 & 0 & 0 \\
0 & \sigma_I & \sigma_I & 0 \\
0 & 0 & 0 & \sigma_S
\end{bmatrix}.$$
Then, we have:

\[
\begin{bmatrix}
\frac{\lambda_r}{\gamma} + \left(\frac{1}{\gamma} - 1\right) a_r \beta_r (T - t) + l(s) \frac{p_r}{\gamma} \\
\frac{\lambda_i}{\gamma} + l(s) \frac{p_i}{\gamma} \\
\frac{\lambda_s}{\gamma} + l(s) \frac{p_s}{\gamma}
\end{bmatrix}
= \Sigma \cdot
\begin{bmatrix}
x_{CB}^{CRRRA(t)} \\
x_{CI}^{CRRRA(t)} \\
x_{CS}^{CRRRA(t)}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

Thus, we can deduce the optimal weights:

\[
\begin{bmatrix}
x_{CB}^{CRRRA(t)} \\
x_{CI}^{CRRRA(t)} \\
x_{CS}^{CRRRA(t)}
\end{bmatrix}
= \left[\Sigma^{-1}\right] \cdot
\begin{bmatrix}
\frac{\lambda_r}{\gamma} + \left(\frac{1}{\gamma} - 1\right) a_r \beta_r (T - t) + l(s) \frac{p_r}{\gamma} \\
\frac{\lambda_i}{\gamma} + l(s) \frac{p_i}{\gamma} \\
\frac{\lambda_s}{\gamma} + l(s) \frac{p_s}{\gamma}
\end{bmatrix}
+ \left(1 - \frac{1}{\gamma}\right) \begin{bmatrix}
0 \\
0 \\
\sigma_f
\end{bmatrix}.
\]

This result can be interpreted as follows:

\[
\begin{bmatrix}
x_{CB}^{CRRRA(t)} \\
x_{CI}^{CRRRA(t)} \\
x_{CS}^{CRRRA(t)}
\end{bmatrix}
= \frac{1}{\gamma} \left[\Sigma^{-1}\right] \cdot
\begin{bmatrix}
\lambda_r + l(s) p_r \\
\lambda_i + l(s) p_i \\
\lambda_s + l(s) p_s
\end{bmatrix}
+ \left(1 - \frac{1}{\gamma}\right) \left[\Sigma^{-1}\right] \cdot
\begin{bmatrix}
-a_r \beta_r (T - t) \\
0 \\
\sigma_f
\end{bmatrix}.
\]

Since \(x_{Bi}^{CRRRA(t)} = 0\), we deduce the equality:

\[
l(t) = - \left(\frac{\nu \mathbb{E}[\zi(t)|\Sigma \mathbb{E}[\nu]]}{\mathbb{E}[\nu]} + (\gamma - 1) \left(0, 1, 0, 0, \frac{\nu \mathbb{E}[\nu]}{\mathbb{E}[\nu]}\right)\right).
\]

### 6.4 Appendix C: Compensating variation for CRRA case

Recall that the function \(G\) is defined by (see 52):

\[
G(z, \nu, t) = \mathbb{E}_\nu \left[ \left[ I_{T \kappa^{(\nu)}(t)}(t) \right]^2 \right].
\]

Note also that, due to the normality of \(I_{T \kappa^{(\nu)}(t)}(t)\), \(ln(G)\) is a quadratic function w.r.t \(\nu\) (\(\nu\) deterministic).

From (55), the CRRA optimal solutions are given by: \((V_0, \tilde{V}_0)\) correspond respectively to the initial amounts invested when there is no constraint and when constraint does exist).
\[ V^{(0)}(T) = V_0 \frac{\left[I_{TK}^{(0)}(T)\right]^{-\frac{1}{i}}}{\mathbb{E}_P\left[\left[I_{TK}^{(0)}(T)\right]^{1-\frac{1}{i}}\right]}, \]

\[ V^{(\lambda)}(T) = \tilde{V}_0 \frac{\left[I_{TK}^{(\lambda)}(T)\right]^{-\frac{1}{i}}}{\mathbb{E}_P\left[\left[I_{TK}^{(\lambda)}(T)\right]^{1-\frac{1}{i}}\right]}, \]

By computing and equalizing the expected utilities of both optimal portfolios, we deduce:

\[
\left[\frac{\tilde{V}_0}{V_0}\right]^{(1-\gamma)} \mathbb{E}_P \left[ \frac{\left[I_{TK}^{(\lambda)}(T)\right]^{-\frac{1}{i}(1-\gamma)}}{\mathbb{E}_P\left[\left[I_{TK}^{(\lambda)}(T)\right]^{1-\frac{1}{i}(1-\gamma)}\right]} \right] = \left[\frac{\tilde{V}_0}{V_0}\right]^{(1-\gamma)} \mathbb{E}_P \left[ \frac{\left[I_{TK}^{(0)}(T)\right]^{-\frac{1}{i}(1-\gamma)}}{\mathbb{E}_P\left[\left[I_{TK}^{(0)}(T)\right]^{1-\frac{1}{i}(1-\gamma)}\right]} \right],
\]

from which we have:

\[
\frac{\tilde{V}_0}{V_0} = \mathbb{E}_P \left[ \frac{\left[I_{TK}^{(0)}(T)\right]^{(1-\gamma)}}{\left[I_{TK}^{(\lambda)}(T)\right]^{(1-\gamma)}} \right]^{\frac{1}{1-\gamma}} = \frac{G((1-\frac{1}{i}), 0, T)}{G((1-\frac{1}{i}), \lambda, T)}^{\frac{1}{1-\gamma}}.
\]