
On the debt capacity of growth and decay options

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On the debt capacity of growth and decay options

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Abstract

This paper focuses on the impact of debt on the optimal policy for investment and hiring, in the light of the theory of real options. We consider a stochastic demand for the product sold by the company. We examine in particular the investment and hiring choices when, in case of enhancement levels of investment and employment, the additional cost is funded in whole or through the issuance of a new bond. We deal with this management problem in the Merton (1974)’s framework. We detail and analyze the optimal strategies. We show in particular that, in the presence of growth option, the value of the firm is the sum of two barrier options, one of "up-and-in call" type, the other of "up-and-out call" type. The barrier corresponds to a threshold of the demand for the product sold by the firm. The novelty of our approach is based on the combination of the optimization of the firm management for a given production function with the valuation of its equity and debt as in Merton (1974). Recall that, in such a case, the equity value is a call option with value of the firm as underlying asset and the amount of the debt to be repaid at maturity as the strike. Our results differ quite significantly from those of the standard framework introduced by Tserlukveich (2008).

Keywords: corporate investment; firing and hiring; stochastic demand; barrier options; real options.

JEL classification: G31, G32, D92.

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1 Introduction

Since the seminal contribution of Merton (1974), the modelling of debt structure has been extensively investigated (see e.g. Leland, 1994a). Hugonnier et al. (2012) study the impact of frictions on the credit market and its implications for the dynamics of the business, its capital structure and default risk. They develop a dynamic model of capital structure taking account of the possibility or not to raise funds on the credit markets. This work belongs to the literature on structural models of dynamic capital with refinancing costs, such as those of Fisher et al. (1989), Leland (1998) and Goldstein et al. (2001). It is shown that usually the refinancing costs generate a rebalancing strategy management more spaced in time resulting in sub-optimality with respect to the targeted objective (see Strebulaev, 2007, for this type of illustrative property). Hackbarth et al. (2006) and Bhamra et al. (2010) propose to integrate macroeconomic variables to better determine the optimal financing policies. Morelec et al. (2012) examine the effect of different agency costs. A growing number of studies examine the financing decisions in the presence of a structure of "roll-over" debt. Such an approach assumes that the replacement of an exogenous fraction of the debt by new debt is done without any cost at every moment (e.g. Leland, 1994b; Leland and Toft, 1996; Hilberink, 2002, Eom et al. 2004; Cheng and Milbradt, 2012; and Schroth et al., 2012). Décamps and Villeneuve (2014) also show that there exists a unique equilibrium in this type of model in which strategies are based on the crossing of a given barrier.

It is not so easy to adequately model the various forms of debt of a firm while getting sufficiently explicit solutions. Two main groups can be distinguished: the first one is based on the existence of a finite horizon at which the debt must be repaid (as in Merton, 1974); the second one claims the existence of an infinite horizon and that the debt can be continuously rolled over (as in Leland, 1994a).\(^1\) The investment in the debt of a company bears the default risk that may suddenly arise. Faced with commitments it can no longer cope, the company may indeed be unable to repay the whole or part of the capital initially borrowed. Such approach called the "structural" approach proposes a modeling of the risk of default of a company.\(^2\) The firm has assets (buildings, production equipment...) funded by the shareholders, but also by issuing bonds. The company may pay dividends to shareholders and distribute coupons to bondholders. However, contractually, in case of failure, shareholders receive the balance of assets (i.e. after deduction of payments due to creditors). Merton (1974) model is in fact fundamentally based on the Black-Scholes-Merton valuation method. In this approach, the value of the share is defined as a call option corresponding to the purchase of the underlying value of the firm and with an exercise price equal to the amount of the debt to be repaid at matur-

\(^1\) See for example Lévyne and Heller (2013) and Navatte (1992, 1998).

\(^2\) An alternative approach, the so-called "intensity" approach, is based on the probability of an event of default, which is directly calibrated. This event of default occurs according to a given probability distribution, for example determined from default spreads quoted on the market (see for example Bielecki and Rutkowski, 2004).
rity. The valuation is done with respect to a risk neutral probability (see next section). The assumptions associated to this model are mainly the fact that financial markets give no arbitrage opportunity, are perfectly liquid, contain a risk-free asset, have no transaction cost and that there is no specific constraint on management strategies. The usual criticism of the Merton’s model is mainly the fact that there is only one maturity and the debt is static. Geske (1977) proposes to generalize the approach of Merton considering that the debt has not a single deadline, which leads to the concept of compound option for derivative assets with underlying the value of the firm. Black and Cox (1976) introduce the possibility for creditors to initiate bankruptcy before the maturity of the debt (i.e. when the value of assets falls below a given threshold). In addition, different types of debt coexist: senior debt, junior debt that starts to be repaid only after full repayment of the senior debt. Other models were also introduced later, as the model of Longstaff and Schwartz (1995) for which the interest rate is stochastic, the model of Brys and de Varenne (1997)...In this paper, we choose to examine the Merton (1974) approach. Therefore, we choose to solve the problems posed by the issuance of a new debt, but for a finite horizon as in Merton (1974), where the initial debt must be repaid at fixed maturity without possibility of renewal on that date.

We consider here a model quite markedly different from the Tserlukevich (2008) framework, to the extent that the debt is based on the issuance of an initial bond that would be "actually" paid for a given maturity. Indeed, it is no longer here a perpetual debt, assumed to be proportional at any time to the value of the firm. In the latter situation, the debt is evaluated at each time \( t \) as equal to the value of the company if, from this date \( t \), the demand becomes constant (\( S_u = S_t \) for any time \( u \) after \( t \)). One implicitly assumes here that the issuer and the debt buyer agree to promote the debt as being equal to the perpetuity of interest rate determined at each time as a function of observed cash flow payments. It follows that, in the absence of growth option, the value of the company, the share value and the debt value are directly proportional (with constant proportionality factors). However, this type of property can be questioned. On the one hand, any potential default is excluded; on the other hand, due to the relationship of proportionality between the fixed value of the company and the value of the debt, this latter one is only considered as a fraction of the value of the firm. It plays somehow the same role as the share issued by the company. This suggests that such assumption corresponds to the issuance of new shares with different rights.

This paper is organized as follows. Section 2 addresses the problem of investment and hiring in the Merton (1974) framework. We detail and analyze the optimal strategies with and without growth option. We show in particular that, in the presence of growth option, the value of the firm is the sum of two barrier options, one corresponding to an "up-and-in call" type, the other to an "up-and-out call." The barrier is equivalent to a threshold of the demand for

\[ S_u = S_t \]

\[ S_u = S_t \]
the product sold by the firm. Section 3 is devoted to the firing and divestment strategy. We analyze this management problem with and without the decay option. We prove that, in the presence of the decay option, the value of the firm is also the sum of two barrier options, one corresponding to a "down-and-out call" type, the other to a conditional "down-and-in call." The barrier corresponds also to another threshold of the demand for the product sold by the firm. We illustrate numerically how the optimal values of firm, equity, debt and growth or decay option depend on both financial and management parameters.

2 Optimal investment and employment policies

2.1 Valuation in the absence of growth option

We consider that the demand $S$ for the product manufactured by the company follows a geometric Brownian motion (GBM) solution of:

$$dS_t = S_t [\mu dt + \sigma dW_t],$$

where $\mu$ and $\sigma$ are the growth rate and the volatility of demand, assumed to be constant. The process $W$ denotes a standard Brownian motion.

The initial investment of the firm level is denoted by $K_0$, and the employment is denoted $L_0$. The production function is a Cobb-Douglas function given by:

$$\pi(S_t, K_0, L_0) = S_t K_0^\alpha L_0^\beta,$$

with $0 < \alpha, \beta < 1$.

The discounted present value of the firm is based on the expectation of future cash flows, which leads us to the following relation:

$$V_t(S_t, K_0, L_0) = E_t \left[ \int_t^\infty e^{-r(1-\tau)(s-t)} S_s K_0^\alpha L_0^\beta ds \right] = \frac{K_0^\alpha L_0^\beta}{r(1-\tau)-\mu} S_t. \quad (1)$$

Proposition 1 The firm's value dynamics is given by:

$$dV_t = V_t [\mu dt + \sigma dW_t],$$

with

$$V_0(S_0, K_0, L_0) = \frac{K_0^\alpha L_0^\beta}{r(1-\tau)-\mu} S_0. \quad (2)$$

Under the theorem of Modigliani and Miller (1958), the equity value $E_t$ of the firm is equal to the discounted present value of the firm to which we subtract the debt $D_t$ incurred to finance the activity. Then we obtain:

$$E_t(S_t, K_0, L_0) = V_t(S_t, K_0, L_0) - D_t.$$
exists a finite horizon $T$ at which a refund $D$ should be performed. Therefore, we adopt the framework considered by Merton (1974). Under these assumptions, once the strategy $(K_0, L_0)$ is fixed, the value of the debt is determined as follows: At maturity, under the terms of seniority in case of firms's bankruptcy, the bond value is given by:

$$B_T(S_T, K_0, L_0, T) = \min[V_T(S_T, K_0, L_0), D].$$

The equity value $E$ is defined as a call option to purchase the underlying value of the company $V$ and with exercise price corresponding to the amount of the debt to be repaid on the horizon $T$ (face value is denoted $D$). We deduce:

**Proposition 2** *(Share value in the absence of the growth option)*

If the manager does not use his growth option (i.e. invested capital and employment levels remain the same), the value of the share is given by: for $t < T$,

$$E_t(S_t, K_0, L_0) = V_t(S_t, K_0, L_0) N(d_1(t)) - De^{-r(T-t)} N(d_2(t)),$$

where $N(.)$ denotes the distribution function of the standard Gaussian distribution and with

$$d_1(t) = \frac{\log \left( \frac{S_t}{D} \right) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}},$$

$$d_2(t) = d_1(t) - \sigma \sqrt{T-t}.$$

As in Merton (1974), we can deduce the bond value at any time.

**Proposition 3** *(Bond value in the absence of the growth option)*

If the manager does not use his growth option, the value of the bond is given by: for $t < T$,

$$B_t(S_t, K_0, L_0) = V_t(S_t, K_0, L_0) - E_t(S_t, K_0, L_0) =$$

$$B_t(S_t, K_0, L_0) = De^{-r(T-t)} \left[ N(h_2(t)) + \frac{1}{d} e^{-r(T-t)} N(h_1(t)) \right],$$

with

$$h_1(t) = - \left[ \frac{(\sigma^2/2)(T-t) - \log [d]}{\sigma \sqrt{T-t}} \right],$$

$$h_2(t) = - \left[ \frac{(\sigma^2/2)(T-t) + \log [d]}{\sigma \sqrt{T-t}} \right],$$

and where $d$ denotes the ratio of debt $De^{-r(T-t)}/V_t(S_t, K_0, L_0)$ based on the face value of the discounted debt.

**Remark 4** We recall that the spread of this bond is defined by :

$$\text{Exp} \left[ -R(t)(T-t) \right] = B_t(S_t, K_0, L_0) / D,$$

which gives us here:

$$R(t)(T-t) - r = \frac{-1}{(T-t)} \log \left[ N(h_2(t)) + \frac{1}{d} e^{-r(T-t)} N(h_1(t)) \right].$$
2.2 Investment and employment with a growth option

We consider now the problem of a firm that has to decide to increase or not its capital $K_0$ to $K_1^+$ and employment level $L_0$ to $L_1^+$, at unknown time $T_1^+$. Then the company searches to maximize the value of its equity $E_{t,t}(S_t, K_0, L_0)$ exercising the growth option holdings.

For such an adjustment, there are transaction costs, denoted here by $C(K_0, K_1^+, L_0, L_1^+)$ and given by:

$$C(K_0, K_1^+, L_0, L_1^+) = p^+(K_1^+ - K_0) + q^+(L_1^+ - L_0).$$

(3)

where $p^+$ (resp. $q^+$) refers to the proportional cost applied when the company rises its capital (resp. its employment level).

This corresponds always to the crossing of a threshold $S^*_\text{max}$ by the market demand. However, if planned, this operation shall be made before the fixed maturity $T$. Therefore, we must take account of this last constraint.

Therefore, $T_1^+$ is the date when the firm decides to invest and hire because the market demand $S$ has reached or exceeded the critical threshold $S^*_\text{max}$.

A new amount equal to the entire cost is financed through the issuance of new bonds. We assume that maturity is still the horizon $T$. We have the equality:

$$D_1 = C^+(K_0, K_1^+, L_0, L_1^+).$$

The value of the firm $V_t$ is given by: if $t < T_1^+$

$$V_t = \mathbb{E}_t \left[ \left( \int_{t}^{T_1^+} e^{-r(1-\tau)(s-t)} K_0^\alpha L_0^\beta S_s ds + \int_{T_1^+}^{\infty} e^{-r(1-\tau)(s-t)} K_1^\alpha L_1^\beta S_s ds \right) \mathbb{I}_{T_1^+ \leq T} \right]$$

$$+ \mathbb{E}_t \left[ \left( \int_{t}^{\infty} e^{-r(1-\tau)(s-t)} K_0^\alpha L_0^\beta S_s ds \right) \mathbb{I}_{T_1^+ > T} \right].$$

(4)

Note that the cost of investment and hiring is fully financed by the issuance of new debt (it is not to be deducted from net worth).

Given the constraint of the horizon $T$, we obtain a more complex formula to describe the optional nature of the investment decision and hiring. Upon issuance of the first bond with $D = \rho K_0 L_0$, we can consider the fact that the manager can or cannot reinvest and hire.
Case 1: ("theoretical" case)

Upon issuance of the first bond $B$, the possibility of the second emission is taken into account. Then it is a single but compound bond. This leads us to the following relationship:

$$B_T = \min \left[ V_T, D + D_1 \right] \mathbb{1}_{T_1^+ \leq T} + \min \left[ V_T, D \right] \mathbb{1}_{T_1^+ > T}. \quad (5)$$

Case 2: ("standard" case)

The first bond (the "senior") $B^{(0)}$ is issued without considering the possible opportunity to issue another one before maturity. The second (the "junior") $B^{(1)}$ will be issued or not depending on market conditions. It will be refunded after full repayment of the first. Thus we obtain:

$$B^{(0)}_T = \min \left[ V_T, D \right],$$

$$B^{(1)}_T = \min \left[ V_T - D, D_1 \right] \mathbb{1}_{T_1^+ \leq T} \mathbb{1}_{V_T > D}.$$ 

We note that, in both situations, the equity value $E_T$ at maturity is given by:

$$E_T = \left[ V_T - (D + D_1) \right] \mathbb{1}_{T_1^+} + \left[ V_T - D \right] \mathbb{1}_{T_1^+ > T}. \quad (6)$$

Recall that $T_1^+$ is the first time of crossing by the demand $S$ of a threshold $S^*$ that has to be determined. The face value $D_1$ of the second bond must also be identified.

**Remark 5** In the absence of specific tax, it is equivalent to assume that the manager allocates costs $C^+(K_0, K_1^+, L_0, L_1^+)$ due to investment and hiring in the following proportions:

$$xC^+(K_0, K_1^+, L_0, L_1^+)$$

form of shrinkage in the value $V$ (equivalent to an "equity financing" strategy) and

$$(1 - x)C^+(K_0, K_1^+, L_0, L_1^+)$$

through the issuance of a new bond of face value

$$D_1 = (1 - x)C^+(K_0, K_1^+, L_0, L_1^+).$$

Indeed, according to the approach developed here, only the value $V_T$ minus the amount of the original debt and costs is important to determine the equity value $E_t$ at any time.
2.2.1 Growth option and equity valuation

The calculation of the equity value \( E_t \) at each time \( t \) during the period \([0, T]\) is performed by observing that it corresponds to the valuation of an "up-and-out call" type (UOC) barrier option with payoff \([V_T - D]^+ \mathbb{1}_{T^+} \) combined with an "up-and-in call" type (UIC) barrier option but conditional. Since we have \( V_t(S_t, K_0, L_0) = \frac{K_0^2 L_0^2 (1-\tau)}{r(1-\tau)-\mu} S_t \), we can deduce that the condition \( S_T > S^* \) is the crossing through the process \( V \) of the value \( V_0^* \) given by:

\[
V_0^* = \frac{K_0^2 L_0^2 (1-\tau)}{r(1-\tau)-\mu} S^*.
\]

We use standard option pricing formulas barrier (e.g., Jeanblanc et al., 2009). We obtain:

Denote by \( C(t, V_t, K, T) \) the value at time \( t \) of the call with underlying \( V_t \) and volatility \( \sigma_V \), exercise price \( K \) and maturity \( T \). This value is given by:

\[
C(t, V_t, K, T) = V_t N(d_1(t)) - K e^{-(1-\tau)(T-t)} N(d_2(t)),
\]

with

\[
d_1(t) = \frac{\log \left( \frac{V_t}{K} \right) + (r(1-\tau) + \sigma_V^2/2)(T-t)}{\sigma_V \sqrt{T-t}},
\]
\[
d_2(t) = d_1(t) - \sigma_V \sqrt{T-t}.
\]

Denote by \( C_d(t, V_t, K, T, \sigma_V) \) the value of the corresponding digital option given by

\[
C_d(t, V_t, K, T) = e^{-(1-\tau)(T-t)} N(d_2(t)).
\]

Set \( \zeta = 1/2 - r(1-\tau)/\sigma_V^2 \).

Recall that the value of the "up-and-out call" (UOC) option with payoff \([V_T - K]^+ \mathbb{1}_{\inf \{t \geq 0 \mid V_t \geq V^*_0 \} > T} \) is given by:

\[
UOC(t, V_t, K, V_0^*, T) =
\]
\[
C(t, V_t, K, T) - C(t, V_t, V_0^*, T) - (V^* - K) C_d(t, V_t, V_0^*, T)
\]
\[
- \left( \frac{V_t}{V_0^*} \right)^{(2\zeta)} \times \left[ C(t, V_0^* / V_t, K, T) - C(t, V_0^* / V_t, V_0^*, T) + (V_0^* - K) C_d(t, V_0^* / V_t, V_0^*, T) \right].
\]
The value of the other option payoff \([V_T - K]^+ \mathbb{1}_{t \geq 0, |V_t| \geq V^*} \leq T\) is more complex, given that the dynamics of value process \(V\) undergoes changes from \(T_1^+\).

**Lemma 6** The density of the first time of reaching the threshold \(V^*\) by the process \(V\) is given by:

\[
f_{T_1^+}(u) = \frac{|\ln [S^*/S_0]/\sigma|}{\sqrt{2\pi} u^{3/2}} \exp \left[-\frac{(\ln [S^*/S_0]/\sigma - (\mu/\sigma - \sigma/2) u)^2}{2u}\right].
\]

**Proof.** Consider the Brownian motion with drift : \(W^{(m)}_t = mt + W_t\), where \(W\) is a standard Brownian motion and \(m\) is a constant. From Result (2.02) on page 295 of Borodin and Salminen (2002), the density of the first hitting time \(H_z\) of threshold \(z\) by the process \(W^{(m)}\) such as \(W^{(m)}_0 = x\) satisfies:

\[
f_{H_{x,z}}(u) = \frac{|z - x|}{\sqrt{2\pi} u^{3/2}} \exp \left[-\frac{(z - x - m u)^2}{2u}\right].
\]

Here, we perform the following identification:

\[
S_t = S_0 \exp \left[(\mu - \sigma^2/2) t + \sigma W_t\right] = S_0 \exp [\sigma ((\mu/\sigma - \sigma/2) t + W_t)].
\]

By asking \(m = (\mu/\sigma - \sigma/2)\), we deduce that the inequality \(S_t \geq S^*(\text{equivalent to } V_t \geq V^*)\) is equivalent to \(W^{(m)}_t \geq \ln [S^*/S_0]/\sigma\). By asking \(x = 0\) and \(z = \ln [S^*/S_0]/\sigma\), we obtain:

\[
f_{T_1^+}(\theta) = f_{H_{x,z}}(u) = \frac{|\ln [S^*/S_0]/\sigma|}{\sqrt{2\pi} u^{3/2}} \exp \left[-\frac{(|\ln [S^*/S_0]/\sigma - (\mu/\sigma - \sigma/2) u|^2}{2u}\right].
\]

In what follows, we illustrate numerically the probability distribution of the first hitting time \(T_1^+\) of threshold \(V^*\). For this numerical example, we consider the following parameter values: \(\sigma = 0.1, \mu = 0.01\).

---

\(4\)This means that we cannot directly use the formula corresponding to the valuation of the "up-and-in call" (UIC) option with payoff

\([V_T - K]^+ \mathbb{1}_{t \geq 0, |V_t| \geq V^*} \leq T\).

Recall that, when the value \(V\) remains the same (before and after \(T_1\)), this one is deduced as follows:

\[UIC(t, V_0, K, V^*, T) = C(t, V_1, K, T) - UOC(t, V_0, K, V^*, T)\].
Figure 1: Probability density function of $T_1$ (investment and hiring).

Table 1: Expectation and probability of being smaller than $T$ (with $T=20$ years)

<table>
<thead>
<tr>
<th>$S^*$ as function of $S_t$</th>
<th>$E[T_1^+]$</th>
<th>$P[T_1^+ \leq T]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^* = 1.2 , S_t$</td>
<td>32</td>
<td>0.74</td>
</tr>
<tr>
<td>$S^* = 1.5 , S_t$</td>
<td>70</td>
<td>0.44</td>
</tr>
<tr>
<td>$S^* = 1.7 , S_t$</td>
<td>90</td>
<td>0.30</td>
</tr>
<tr>
<td>$S^* = 2 , S_t$</td>
<td>116</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Now we keep on the calculation of the value of the other option with payoff $[V_T - K]^+ \mathbb{1}_{\{t \geq 0, V_t \geq V_T\} \leq T}$. Using the properties of conditional expectation, we can write the following relationship: For $t < T_1^+$,

$$e^{-r(T-t)}E_{Q,t} \left[ [V_T - K]^+ \mathbb{1}_{T_1^+ \leq T} \right] =$$

$$E_{Q,t} \left[ e^{-r(T^+_1 - t)} e^{-r(T-T_1^+)} \mathbb{1}_{T_1^+ \leq T} E_{Q,T_1^+} \left[ [V_T - K]^+ \right] \right]. \quad (8)$$

with $e^{-r(T-T_1^+)}E_{Q,T_1^+} \left[ [V_T - K]^+ \right]$ corresponds to a formula of the call in the
Black and Scholes framework. Indeed, we have:

\[ e^{-r(T-T_1)} \mathbb{E}_{Q,T_1^+} \left[ (V_T - K)^+ \right] = C(T_1^+, V_0^*, K, T). \]

Using the properties of independent increments of Brownian motion, we deduce that the density of the first hitting time of the threshold \( V_0^* \) by the process \( V \) conditionally to information held at the time \( t \) is equal to:

\[
f_{t,T_1^+}(\theta) = \frac{\ln [V_0^*/V_t] / \sigma}{\sqrt{2\pi \theta^{3/2}}} \exp \left[ -\frac{(\ln [V_0^*/V_t] / \sigma - (\mu / \sigma - \sigma / 2) \theta)^2}{2\theta} \right]. \tag{9}
\]

Now introduce the value \( V_1^* \) that matches the value of the firm at time \( T_1^+ \) where the demand \( S \) crosses the threshold \( S^* \) but for a level of investment \( K_1^+ \) and employment \( L_1^+ \). We obtain:

\[
V_1^* = \frac{K_1^{+}\alpha L_1^{+\beta} (1 - \tau)}{\tau (1 - \tau) - \mu} S^*. \tag{10}
\]

So we deduce:

**Proposition 7** The value of the "up-and-in call" option (UIC) in the presence of a jump at time \( T_1^+ \) in the dynamics of the process value \( V \) is given by:

\[
e^{-r(1-\tau)(T-t)} \mathbb{E}_{Q,t} \left[ (V_T - (D + D_1))^{\tau} I_{T_1^+ \leq T} \right] \tag{11}
\]

\[= \int_0^{T-t} e^{-r(1-\tau)\theta} C(\theta + t, V_1^*, D + D_1, T) f_{t,T_1^+}(\theta)d\theta. \]

**Remark 8** Formula (11) is more general than that of Shackleton and Wojakowski (2007). Here we must take account of the uncertainty surrounding the date of issuance of the option.

In what follows, we use the function \( UICc(t, V_0, D + D_1, V_1^*, T) \) which corresponds to the valuation of the previous barrier "up-and-in call" type (UIC) option taking account of the impact beared by the firm value when reinvesting and hiring.

**Proposition 9** (Value of the share in the presence of growth option)

If the manager uses his growth option, the equity value \( E_t \) is given by: For \( t < T_1 \),

\[ E_t = UICc(t, V_1^*, D + D_1, V_0^*, T) + UOC(t, V_t, D, V_0^*, T). \tag{12} \]

For \( t < \text{Min}(T, T_1^+) \), maximizing the value of the share is then based on two steps:

- First, for a given threshold value \( S^* \), find the optimal levels \( K_1^+ \) and \( L_1^+ \);
- Second, find the optimal threshold value \( S^* \) by using the optimal functions \( K_1^+(S^*) \) and \( L_1^+(S^*) \).
We have to consider the maximization (relative to variables \((K_1^+, L_1^+)\) according to the threshold \(S^*\)) of the term

\[
UICc(t, V_1^*, D + D_1, T) = \int_0^{T-t} e^{-r(1-r)\theta} C(\theta + t, V_1^*, D + D_1, T) f_{t, T_1^+}(\theta) d\theta.
\]

More specifically, we need to solve the following problem:

\[
\text{Max} \quad K_1^+, L_1^+ \left[ UICc(t, V_1^*, D + D_1, T) \right]
\]

with

\[
D_1 = p^+(K_1^+ - K_0) + q^+(L_1^+ - L_0) \quad \text{and} \quad V_1^* = \frac{(1 - \tau)S^*}{r(1 - \tau) - \mu} K_1^+ L_1^+. \]

The first order conditions are as follows:

\[
\frac{\partial UICc(t, V_1^*, D + D_1, T)}{\partial K_1^+} = \int_0^{T-t} e^{-r(1-r)\theta} \frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial K_1^+} f_{t, T_1^+}(\theta) d\theta.
\]

Using sensitivities (the "Greeks") of the standard call, we deduce:

\[
\frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial K_1^+} = \frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial V} \frac{\partial V}{\partial K_1^+} + \frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial D} \frac{\partial D}{\partial K_1^+}.
\]

Similarly, we obtain:

\[
\frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial L_1^+} = \frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial V} \frac{\partial V}{\partial L_1^+} + \frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial D} \frac{\partial D}{\partial L_1^+}.
\]
Remark 10 We note that the optimal solutions obtained by solving the system
\[
\frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial K_1^+} = 0 \\
\frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial L_1^+} = 0,
\]
still satisfy the relation:
\[
\frac{K_1^+}{L_1^+} = \frac{\alpha q^+}{\beta p^+}. \quad (13)
\]
We deduce that:
\[
L_1^+ = K_1^+ \left(\frac{\alpha q^+}{\beta p^+}\right)^{-1} \text{ and } V_1^* = \frac{(1 - \tau)S^*}{r(1 - \tau) - \mu} K_1^{+\alpha + \beta} \left(\frac{\alpha q^+}{\beta p^+}\right)^{-\beta}
\]
This leads us to solve for example the following equation in order to determine $K_1^+$ based on the threshold $S^*$:
\[
\int_0^{T-t} e^{-r(1-\tau)\theta} \left(\frac{\partial C(\theta + t, V_1^*, D + D_1, T)}{\partial K_1^+}\right) f_{K_1^+}(\theta)d\theta = 0,
\]
where
\[
d_1(\theta + t, V_1^*, D + D_1, T) = \frac{\log \left(\frac{V_1^*}{D + D_1}\right) + (r(1 - \tau) + \sigma^2/2)(T - \theta - t)}{\sigma \sqrt{T - \theta - t}},
\]
\[
d_2(\theta + t, V_1^*, D + D_1, T) = \frac{d_1(\theta + t, V_1^*, D + D_1, T) - \sigma \sqrt{T - \theta - t}}{\sigma \sqrt{T - \theta - t}}.
\]
with:
\[
V_1^* = \frac{(1 - \tau)S^*}{r(1 - \tau) - \mu} K_1^{+\alpha + \beta} \left(\frac{\alpha q^+}{\beta p^+}\right)^{-\beta},
\]
and
\[
D_1 = p^+(K_1^+ - K_0) + q^+(K_1^+ \left(\frac{\alpha q^+}{\beta p^+}\right)^{-1} - L_0).
\]
In what follows, we illustrate numerically the optimal investment and hiring solutions. We consider the following basic parameter values:
\[
\alpha = 0.2, \beta = 0.3, K_0 = 100, L_0 = 100, p^+ = 1.2, q^+ = 1.1,
\]
\[
\sigma = 0.1, \mu = 0.01, r = 0.05, \tau = 0.1, S_0 = 1.
\]
At first, we show how changes the value of the "up-and-in call" option $UICc(t, V_1^*, D + D_1, T)$ based on investment $K_1^+$ and trigger threshold $S^*$. The
value of the employment level corresponds to that one given by the formula optimality (13). Note that this option aggregates somehow respectively increasing and decreasing functions in \( S^* \). The value of the call being an increasing function of the value of the underlying is, as a function of this first component, an increasing function of the threshold \( S^* \) through the expression of \( V_1^* \). But it is also a decreasing function of the exercise price and therefore, as a function of the second component, a decreasing function of the threshold \( S^* \) via the expression of \( D_1 \) if growth \( K_1^+ (S^*) \). Note that the function \( \mathbb{I}_{T^* \leq T} \) decreases with respect to the threshold \( S^* \). Indeed, the higher the threshold, the lower the probability of the crossing. These effects are partially offset which explains the appearance of the internal maximum.

From Figure (2)(with \( S^* \) x-axis and y-axis \( K_1^+ \)), we observe that there is an interior maximum \( S^* \) for \( K_1^+ \) given level. For the digital case presented, it is around \( S^* = 1.015\% \times S_0 \). Here, a slight increase in the price of the product sold triggers investment.

![Option value as function of \( K_1 \) and \( S^* \)](image1)

![Option value as function of \( K_1 \) and \( S^* \)](image2)

Figure 2: "up-and-in call" option value.
We now analyze in particular the role of the level of initial debt $D$.

Figure 3: "up-and-in call" option value as function of the initial debt.

Previous figures show that the value of the "up-and-in call" option is a decreasing function with respect to the value of the original debt $D$. This is due to the fact that this debt is here an exercise price of a call. However, as a function of the investment level $K_1$, the sense of monotony can be reversed: For relative standard values of the initial debt $D$, the increase in $K_1$ has for main effect to increase the total debt maturing and therefore "works against" the "up-and-in call." For high values of the initial debt $D$, the increase in $K_1$ has smaller impact on the percentage of the total debt. Therefore, it is the other aspect that prevails, namely an increase in the value of the firm at maturity, favoring the probability of deal to repay the total debt.

We now study the second step of the optimization problem of the equity value. Recall that, according to the relationship (12), it is given by $E_t = \ldots$
\( UICC(t, V_1^*, D + D_1, T) + UOC(t, V_t, D, V_0^*, T) \). To do this, we use the previous result for the determination of optimal investment and hiring based on the threshold of the demand \( S^* \). Note that the "up-and-in call" option is a decreasing function on the trigger \( S^* \) while the "up-and-out call" option (UOC) is increasing in \( S^* \) (due to the fact that the payment \([V_T - D]^+\) here does not depend on the threshold \( S^* \) and the function \( \mathbb{1}_{T^+ > T} \) is decreasing in \( S^* \)). The "compensation" of these two terms justifies the potential existence of an optimal threshold \( S^* \).

**Remark 11** The value of the UOC option is equal to zero when the threshold \( V^* \) (corresponding to the threshold of the demand \( S^* \)) is smaller than the initial debt level \( D \) (which here is actually the total debt to be repaid at maturity). We deduce that, for maximizing the UOC option, the optimal threshold of the application \( S^* \) must be greater than or equal to the value \( S_D \) such that \( V(S_D) = D \) i.e \( S_D = D \left( \frac{K_0^+ L_0^+ (1-\tau)}{r(1-\tau)-\mu} \right)^{-1} \) from the relationship (1). This means that at the time we proceed to investment and hiring, if the value of the UOC option prevails in share value, then the value of the company must be such that a full refund of the original debt would be possible at that time.

As shown in Equation (12), the optimal crossing threshold \( S^* \) can be obtained by triggering investment and hiring (UICC impact option) or is reached at infinity if the UOC option dominates the total share value of the firm. In this case, the optimal policy is simply to maintain the same levels and hiring \((K^+_1 = K_0 \) and \( L^+_1 = L_0 \)). Comparison of the two optima is of course a function of various parameters of the model. Too high costs discourage such investment and hiring (see Figure 4). Figure (4.1) corresponds to investment and hiring, while Figure (4.2) corresponds to the statu quo. In the latter case, the value of the share is obtained asymptotically but convergence towards this value is very fast. Indeed, from a certain threshold level \( S^* \), the probability of crossing is almost equal to zero.

![Figure 4: Equity value with respect to costs.](image)

1. Equity value for basic costs
2. Equity value for high costs
Now we analyze the role of the initial debt level $D$ on the value of the shares of the company. As shown in Figure (5), an increase of the level of initial debt has the effect of reducing the interest to reinvest and hire. However, an increase of investment and hiring give hope of future higher cash flows. We note here that, for a lower debt level $(K_0 + L_0)$ (see Figures (5.1) and (5.2), the exercise of the growth option is "progressively" wise. For large initial debt (see Figures (5.3) and (5.4), the risk of increasing the overall debt is offset by the hope to generate superior cash flows. But this requires to consider very high demand thresholds $S^*$. 

![Figure 5: Equity value as function of debt](image)

1. Equity value for $D = 0$
2. Equity value for $D = K_0 + L_0$
3. Equity value for $D = 2 (K_0 + L_0)$
4. Equity value for $D = 5 (K_0 + L_0)$

**Proposition 12 (Value of the growth option)**

The value of the growth option $Go(S_t, K_0, L_0, D)$ is given by: For $t < T_1$,

$$Go(S_t, K_0, L_0, D) = UICc(t, V_1^*, D + D_1, T) + UOC(t, V_t, D, V_0^*, T) - C(t, V_t, D, T).$$  \(14\)
In what follows, we illustrate the value of the growth option associated with the investment opportunity and employment. When the values are negative, the threshold levels $S^*$ and investment $K_1$ are not optimal. When the values are positive, the manager prefers to invest and hire. We find the effect of the original debt: Under the basic parameter values, when the original debt is low, the manager chooses to invest and hire and when the original debt is relatively high, the manager chooses to invest and hire but for high threshold levels of demand. For a large debt, the manager invests and recruits for high thresholds but relatively limited levels of investment and hiring compared to a situation where the original debt is lower (note that the value of the share option without growth is very low in this case).

![Figure 6: Growth option value as function of debt](image_url)

1. Growth option value for $D = 0$
2. Growth option value for $D = K_0 + L_0$
3. Growth option value for $D = 2(K_0 + L_0)$
4. Growth option value for $D = 5(K_0 + L_0)$
Now we examine the impact of a sharp increase in volatility of demand for the product sold by the firm. As shown in Figure (7), the increase in volatility has not simply the effect to increase the value of the option growth (in the following figures, volatility is doubled compared to the base case). It can also cause a change in optimal strategy, as shown in Figures (7.1) and (7.2). Indeed, in the absence of growth option, the increase in volatility leads to a reduction in the equity value (remember here it is a call to the underlying value of the firm). The manager will therefore try to compensate for this by focusing on future rising cash flows and by increasing his investment and employment levels (the other parameters being fixed elsewhere). However, note that this difference is lower for higher levels of the original debt.

Figure 7: Growth option value as function of volatility
2.2.2 Valuation of the bond part

Recall that we can distinguish two cases:

**Case 1 ("theoretical"):**

Upon issuance of the first bond \( B \), the possibility of the second issuance is taken into account. Then it is a single bond but based on payoff given \( \beta \):

\[
B_{T;T} = \min \left[ V_T, D + D_1 \right] I_{T \leq T} + \min \left[ V_T, D \right] I_{T > T}. \tag{15}
\]

For this first case, we directly obtain the pricing formula from the equality \( B_{t;T} = V_t - E_t \).

**Proposition 13 (Value of the bond in the presence of growth option)**

If the manager uses his growth option, the value of the bond \( B_{t;T} \) is given by: For \( t < T \),

\[
B_{t;T} = V_t - UICc(t, V_1^*, D + D_1, T) - UOC(t, V_t, D, V_0^*, T). \tag{16}
\]

**Case 2 ("standard"):**

The first bond (the "senior") \( B^{(0)} \) is issued without considering the possible opportunity to issue another before maturity. The second (the "junior") \( B^{(1)} \) will be issued or not depending on market conditions, i.e., depending on the fact that the manager uses or not his option growth. It will be refunded after full repayment of the first one. Thus we obtain:

\[
B^{(0)}_T = \min \left[ V_T, D \right], \quad B^{(1)}_T = \min \left[ V_T - D, D_1 \right] I_{T \leq T}.
\]

**Proposition 14 (Value of senior and junior bonds in the presence of growth option)**

If the manager uses its growth option, the value of the senior obligation \( B^{(0)}_{t;T} \) is given by: For \( t < T \),

\[
B^{(0)}_{t;T} = D e^{-r(1-\tau)(T-t)} - UIPC(t, V_1^*, D + D_1, T) - UOP(t, V_t, D, V_0^*, T). \tag{17}
\]

The value of the junior obligation \( B^{(1)}_{t;T} \) is given by: For \( t < T \),

\[
B^{(1)}_{t;T} = V_t - B^{(0)}_{t;T} - E_t. \tag{18}
\]

**Proof.** To assess the senior bond, we note that:

\[
B^{(0)}_T = \min \left[ V_T, D \right] = D - \max \left[ D - V_T, 0 \right].
\]

However, the value of the firm at maturity is given by:

\[
V_T = V_T I_{T \leq T} + V_T I_{T > T}. \tag{19}
\]
We deduce:

\[ B_T^{(0)} = D - Max [D - V_T, 0] \mathbb{I}_{T_1^+ \leq T} - Max [D - V_T, 0] \mathbb{I}_{T_1^+ > T}. \]

Recall that when the manager decides to invest and hire at time \( T_1^+ \leq T \), the value of the firm just after this time is \( V_T^* = \frac{(1-\gamma)S}{\gamma(1-\gamma) - \mu} K_1^+ \alpha L_1^+ \).

As for the calculation of the value of the share, we must consider the probability distribution of the random time \( T_1^+ \) to value the option \( Max [D - V_T, 0] \mathbb{I}_{T_1^+ \leq T} \).

This is given by the function \( UIP_c(t, V_0, D, V_T^*, T) \) which corresponds to the valuation of the "up-and-in could" type (UIP) barrier option taking account of the impact suffered by the value of the company on the reinvestment and hiring. We obtain:

\[ UIP_c(t, V_T^*, D, T) = \int_0^{T-t} e^{-r(1-\gamma)\theta} P(\theta + t, V_T^*, D, T) f_{t,T_1^+}(\theta) d\theta. \]

The value of the option \( Max [D - V_T, 0] \mathbb{I}_{T_1^+ > T} \) in turn corresponds to the "up-and-out put" option (UOP) given by:

\[ UOP(t, V_t, D, V_0^*, T) = UOC(t, V_t, D, V_0^*, T) - e^{-r(1-\gamma)(T-t)} E_Q [(V_T - D) \mathbb{I}_{T_1^+ > T}], \]

where the value \( UOC(t, V_t, K, V_0^*, T) \) is given in the relation (7).

To assess the junior bond, we note that:

\[ B_T^{(1)} = V_T - B_T^{(0)} - E_T. \]

We deduce:

\[ B_t^{(1)} = V_t - B_t^{(0)} - E_t. \]

\[ \blacksquare \]

**Remark 15** To assess the junior obligation, we can also note that:

\[ B_T^{(1)} = (D_1 - Max [D + D_1 - V_T, 0]) \mathbb{I}_{T_1^+ \leq T}, \]

from which we deduce:

\[ B_t^{(1)} = \int_0^{T-t} e^{-r(1-\gamma)\theta} [D_1 - P(\theta + t, V_T^*, D + D_1, T)] f_{t,T_1^+}(\theta) d\theta. \]

**Remark 16** Recall that the value of the option \( UOP \) is given by:

\[ UOP(t, V_t, D, V_0^*, T) = \]

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Case (i): If $V_0^* > D$,

$$
V_t \left( -N(-d_1(t, V_t, D, T, \sigma)) + \left( \frac{V_t^*}{V_t} \right)^{(1+2r/\sigma^2)} N(-d_1(t, V_0^*, DV_t, T, \sigma)) \right) \\
-De^{-r(1-\gamma)(T-t)} \left( -N(-d_2(t, V_t, D, T, \sigma)) + \left( \frac{V_t^*}{V_t^*} \right)^{(1+2r/\sigma^2)} N(-d_2(t, V_0^*, DV_t, T, \sigma)) \right)
$$

Case (ii): If $V_0^* < D$,

$$
V_t \left( -N(-d_1(t, V_t, V_0^*, T, \sigma)) + \left( \frac{V_t^*}{V_t} \right)^{(1+2r/\sigma^2)} N(-d_1(t, V_0^*, V_t, T, \sigma)) \right) \\
-De^{-r(1-\gamma)(T-t)} \left( -N(-d_2(t, V_t, V_0^*, T, \sigma)) + \left( \frac{V_t^*}{V_t^*} \right)^{(1+2r/\sigma^2)} N(-d_2(t, V_0^*, V_t, T, \sigma)) \right)
$$

Finally, we deduce the values of debt ratios:

**Case 1 ("theoretical")**:

Upon issuance of the first bond $B$, the possibility of the second emission is counted. Then it is a single bond based on the following payoff:

$$
B_{T;T} = \text{Min} [V_T, D + D_1] I_{T^+ \leq T} + \text{Min} [V_T, D] I_{T^+ > T}.
$$

For this first case, we directly obtain the pricing formula from the equality $B_{t,T} = V_t - E_t$.

**Proposition 17 (Value of the debt ratio in the presence of growth option)**

In both cases, the debt ratio is given by:

$$
R_t = \frac{V_t - UICc(t, V_t^*, D + D_1, T) - UOC(t, V_t, D, V_0^*, T)}{V_t}.
$$

In what follows, we illustrate the behavior of the debt ratio. As shown in Figure (8), the debt ratio is minimal when the share value is maximum. It is of course increasing with respect to the value of the original debt $D$. 

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1. Debt ratio for $D = 0$

2. Debt ratio for $D = K_0 + L_0$

3. Debt ratio for $D = 2(K_0 + L_0)$

4. Debt ratio for $D = 5(K_0 + L_0)$

Figure 8: Debt ratio
3 Divestment and Firing (decay option)

Suppose now that the firm holds the option to reduce at the same time and at once his investment and employment levels. We prove in this case that the optimal strategy is to exercise an option to decrease, which corresponds somewhat to the reduction of the size of the project.

Recall that transaction costs \( C(K_0, K_1, L_0, L_1) \) are negative here because of the decrease in investment, but also because the company has a smaller payroll. Therefore we have:

\[
C(K_0, K_1, L_0, L_1) = p^-(K_1^+ - K_0) + q^- (L_1^+ - L_0) < 0.
\]

The value \( V_T \) of the firm at maturity is given by:

\[
V_T = \left( \int_0^{T_1^-} e^{-(1 - \tau)(s-t)} K_0^\alpha L_0^\beta S_s ds + \int_{T_1^-}^T e^{-(1 - \tau)(s-t)} K_1^- L_1^- S_s ds \right) 1_{T_1^- \leq T} + \left( \int_0^T e^{-(1 - \tau)(s-t)} K_0^\alpha L_0^\beta S_s ds \right) 1_{T_1^- > T}.
\]

Note that here disinvestment costs and dismissal being negative, they do not correspond to the issuance of new debt but will have a gain that is added to net worth.

Given the constraint of the horizon \( T \), we obtain a more complex formula to describe the optional nature of the decision to disinvest and fire.

We note that the value \( E_T \) of the action at maturity is given by:

\[
E_T = [V_T - D - C(K_0, K_1, L_0, L_1)]^+ 1_{T_1^- \leq T} + [V_T - D]^+ 1_{T_1^- > T}.
\]

Recall that \( T_1^- \) is the first moment of crossing by the demand of a lower threshold \( S_{min}^* \) to be determined.

3.1 Share value and the decay option

The calculation of the equity value \( E_t \) at any time \( t \) in the period \([0, T]\) is performed by using valuation results of "down-and-out call" barrier option type (UOC) with payoff \([V_T - D]^+ 1_{T_1^- > T}\). Since \( V_t(S_t, K_0, L_0) = \frac{K_0^\alpha L_0^\beta}{r(1 - \mu) - \mu} S_t \), we can deduce that the condition \( S_T < S_{min}^* \) is the crossing through the process \( V \) of the value \( V_{min,0}^* \) given by:

\[
V_{\text{min},0}^* = \frac{K_0^\alpha L_0^\beta(1 - \tau)}{r(1 - \mu) - \mu} S_{\text{min}}^*.
\]
Using standard formulas for valuation of barrier options (see Jeanblanc et al., 2009, page 173), we obtain:

Recall that the value of the "down-and-out call" option (DOC) with payoff $[V_T - K]^+ \mathbb{I}_{\{t\geq 0|V_t < V_{\text{min},0}\}} > T$ is given by:

$$DOC(t, V_t, K, V_{\text{min},0}; T) = C(t, V_t, K, T) - \left( \frac{V_t}{V_{\text{min},0}} \right)^{(-2r/\sigma^2)} C(t, V_{\text{min},0}, KV_t/V_{\text{min},0}; T).$$

As the process value $V$ is modified at time $T_1^-$, the value of the other option $[V_T - K]^+ \mathbb{I}_{\{t\geq 0|V_t < V_{\text{min},0}\}} \leq T$ is more difficult to express. However, we can determine it using the distribution of $T_1^-$. 

**Lemma 18** The density of the first hitting time threshold $V_{\text{min},0}^*$ by the process $V$ is given by:

$$f_{T_1^-}(u) = \frac{\ln [S_{\text{min}}^*/S_0] / \sigma}{\sqrt{2\pi}u^{3/2}} \exp \left[ - \frac{(\ln [S_{\text{min}}^*/S_0] / \sigma - (\mu/\sigma - \sigma/2) u)^2}{2u} \right].$$

<table>
<thead>
<tr>
<th>$S_{\text{min}}^*$ as function of $S_t$</th>
<th>$\mathbb{E}[T_1^-]$</th>
<th>$\mathbb{P}[T_1^- \leq T]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^* = 0.9 S_t$</td>
<td>6</td>
<td>0.77</td>
</tr>
<tr>
<td>$S^* = 0.7 S_t$</td>
<td>14</td>
<td>0.35</td>
</tr>
<tr>
<td>$S^* = 0.5 S_t$</td>
<td>13</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Now we keep on the calculation of the value of the other option $[V_T - K]^+ \mathbb{I}_{\{t\geq 0|V_t \leq V_{\text{min},0}^*\}} \leq T$.

Using the properties of conditional expectation, we can assert the following relation: For $t < T_1^-$,

$$e^{-r(T-t)} \mathbb{E}_{Q,t}[V_T - K]^+ \mathbb{I}_{T_1^- \leq T} =$$

$$\mathbb{E}_{Q,t}[e^{-r(T_1^- - t)} e^{-r(T-T_1^-)} \mathbb{I}_{T_1^- \leq T} \mathbb{E}_{Q,T_1^-}[V_T - K]^+]].$$

(24)

Note that $e^{-r(T-T_1^-)} \mathbb{E}_{Q,T_1^-}[V_T - K]^+$ corresponds to a formula of the call in the Black and Scholes. Indeed, we have:

$$e^{-r(T-T_1^-)} \mathbb{E}_{Q,T_1^-}[V_T - K]^+ = C(T_1^-, V_{\text{min},0}^*, K, T).$$

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Using the properties of independent increments of the Brownian motion, we deduce that the density of the first hitting time threshold \( V_{\min,0}^* \) by the process \( V \) conditional on information held at time \( t \) is equal to:

\[
f_{t, T^{-}_1}(\theta) = \frac{\ln [V_{\min,0}^*/V_t]}{\sqrt{2\pi\theta^3/2}} \exp \left[ - \frac{\left( \ln [V_{\min,0}^*/V_t] / \sigma - (\mu/\sigma - \sigma/2) \theta \right)^2}{2\theta} \right].
\] (25)

Now we introduce the value \( V_{\min,1}^* \) which corresponds to the value of the firm at time \( T^{-}_1 \) where demand \( S \) crosses the threshold \( S_{\min}^* \) but for a level of investment \( K_1^- \) and employment \( L_1^- \). We have:

\[
V_{\min,1}^* = \frac{K_1^- \beta (1 - \tau)}{r (1 - \tau) - \mu} S_{\min}^*.
\] (26)

Then, we deduce:

**Proposition 19** The value of the "down-and-in call" option (DIC) in the presence of a jump at time \( T^+_1 \) in the dynamics of process value \( V \) is given by

\[
\begin{align*}
\exp &-r(1-\tau)(T-t)E_{Q,t} \left[ \left( V_T - D - C(K_0, K_1^-, L_0, L_1^-) \right)^+ \mathbb{1}_{T^-_1 < T} \right] \\
&= \int_0^{T-t} \exp -r(1-\tau)\theta C(\theta + t, V_{\min,1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T) f_{t, T^{-}_1}(\theta) d\theta.
\end{align*}
\] (27)

**Remark 20** As in the previous case corresponding to the growth option, formula (27) is more general than that of Shackleton and Wojakowski (2007) case. Here we must still take account of the uncertainty surrounding the date of issue of this option (DIC).

In what follows, we use the function \( DIC_c(t, V_{\min,1}^*, D + D_1, V_1^*, T) \) which corresponds to the valuation of the previous "down-and-in call" type (UIC) barrier option taking account of the impact on the firm value during the divestment and firing.

**Proposition 21** (Value of the share in the presence of option decay)

If the manager uses his option to decline, the value of the equity \( E_t \) is given by: For \( t < T^-_1 \),

\[
E_t = DIC_c(t, V_{\min,1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T) + DOC(t, V_t, D, V_0^*, T).
\] (28)

For \( t < \text{Min} \left( T, T^-_1 \right) \), we begin by maximizing the following term (with respect to variables \( (K_1^-, L_1^-) \) and according to a given threshold \( S_{\min}^* \)):

\[
\begin{align*}
&DIC_c(t, V_{\min,1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T) = \\
&\int_0^{T-t} \exp -r(1-\tau)\theta C(\theta + t, V_{\min,1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T) f_{t, T^{-}_1}(\theta) d\theta.
\end{align*}
\]
We need to solve the following problem:

$$\max_{K_1^-, L_1^-} \left[ DICc(t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T) \right]$$

with

$$C(K_0, K_1^-, L_0, L_1^-) = p^-(K_1^- - K_0) + q^- (L_1^- - L_0)$$

and

$$V^*_{\min,1} = \frac{(1 - \tau) S^*}{r(1 - \tau) - \mu} K_1^{-\alpha} L_1^{-\beta}.$$

The first order conditions are as follows:

$$\frac{\partial DICc(t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial K_1^-} =$$

$$\int_0^{T-t} e^{-r(1-\tau)\theta} \frac{\partial C(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial K_1^-} f_{t,T_1^-}(\theta) d\theta.$$

Using sensitivities (the "Greeks") of the standard call, we deduce:

$$\frac{\partial C(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial K_1^-} =$$

$$\frac{\partial C(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial V} \frac{\partial V^*_{\min,1}}{\partial K_1^-}$$

$$+ \frac{\partial C(\theta + t, V^*_{\min,1}, D + D_1, T)}{\partial D} \frac{\partial [D + C(K_0, K_1^-, L_0, L_1^-)]}{\partial K_1^-}$$

$$\frac{\partial C(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial L_1^-} =$$

$$\frac{\partial C(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial V} \frac{\partial V^*_{\min,1}}{\partial L_1^-}$$

$$+ \frac{\partial C(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial D} \frac{\partial [D + C(K_0, K_1^-, L_0, L_1^-)]}{\partial L_1^-}$$

$$= N(d_1(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)) \frac{\alpha K_1^{-(\alpha - 1)} L_1^{-\beta} (1 - \tau)}{r(1 - \tau) - \mu} S^*_{\min}$$

$$- e^{-r(1-\tau)(T-\theta-t)} N(d_2(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)) p^-.$$

Similarly, we obtain:

$$= N(d_1(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)) \frac{\beta K_1^{-\alpha} L_1^{-\beta - 1} (1 - \tau)}{r(1 - \tau) - \mu} S^*_{\min}$$

$$- e^{-r(1-\tau)(T-\theta-t)} N(d_2(\theta + t, V^*_{\min,1}, D + C(K_0, K_1^-, L_0, L_1^-), T)) q^-.$$
Remark 22 We note that the optimal solutions obtained by solving the system

\[
\frac{\partial C(\theta + t, V_{\text{min}, 1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial K_1^-} = 0
\]
\[
\frac{\partial C(\theta + t, V_{\text{min}, 1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial L_1^-} = 0,
\]

still satisfy the relation:

\[
\frac{K_1^-}{L_1^-} = \frac{\alpha q^-}{\beta p^-}.
\]

We deduce:

\[
L_1^- = K_1^- \left(\frac{\alpha q^-}{\beta p^-}\right)^{-1} \quad \text{et} \quad V_{\text{min}, 1}^* = \frac{(1 - \tau)S_{\text{min}}^*}{r(1 - \tau) - \mu} K_1^-(\alpha + \beta) \left(\frac{\alpha q^-}{\beta p^-}\right)^{-\beta}.
\]

This leads us to solve the following equation in order to determine $K_1^-$ according to the threshold $S_{\text{min}}^*$:

\[
\int_0^{T-t} e^{-r(1-\tau)\theta} \left(\frac{\partial C(\theta + t, V_{\text{min}, 1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T)}{\partial K_1^-}\right) f_{t, T_t}(\theta)d\theta = 0,
\]

where

\[
d_1(\theta + t, V_{\text{min}, 1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T) = \frac{\log \left(\frac{V_{\text{min}, 1}^*}{D + C(K_0, K_1^-, L_0, L_1^-), T}\right)}{\sigma \sqrt{T - \theta - t}},
\]

and

\[
d_2(\theta + t, V_{\text{min}, 1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T) = d_1(\theta + t, V_{\text{min}, 1}^*, D + C(K_0, K_1^-, L_0, L_1^-), T) - \sigma \sqrt{T - \theta - t}.
\]

with :

\[
V_{\text{min}, 1}^* = \frac{(1 - \tau)S_{\text{min}}^*}{r(1 - \tau) - \mu} K_1^-(\alpha + \beta) \left(\frac{\alpha q^-}{\beta p^-}\right)^{-\beta},
\]

and

\[
C(K_0, K_1^-, L_0, L_1^-) = p^-(K_1^- - K_0) + q^-(K_1^- \left(\frac{\alpha q^-}{\beta p^-}\right)^{-1} - L_0).
\]
3.2 Numerical illustrations

In what follows, we illustrate numerically optimal solutions of divestment and fire. As for the other case, we consider the following basic parameters:

\[
\begin{align*}
\alpha &= 0.2, \beta = 0.3, K_0 = 100, L_0 = 100, p^- = 0.5, q^- = 0.5, \\
\sigma &= 0.1, \mu = 0.01, r = 0.05, \tau = 0.1, S_0 = 1.
\end{align*}
\]

At first, we show how the value of the "down-and-in call"

\[DICc(t, V_{\text{min},1}, D + C(K_0, K_1^-, L_0, L_1^-), T)\]

evolves according to investment \(K_1^-\) and trigger \(S_{\text{min}}^*\). The value of the redundancy level is that given by the optimality formula (13).

Note that this option is based on functions with opposite monotonies with respect to \(S_{\text{min}}^*\). The value of the call is an increasing function of the value of its underlying so, for this component, increasing the threshold in \(S_{\text{min}}^*\) via the expression of \(V_{\text{min},1}\). But it is decreasing with respect to the exercise price, so for this second component can be a decreasing function of the threshold \(S_{\text{min}}^*\) via \(C(K_0, K_1^-, L_0, L_1^-)\) if \(K_1^-(S_{\text{min}}^*)\) and \(L_1^-(S_{\text{min}}^*)\) are increasing in \(S_{\text{min}}^*\).

Note that the function \(\mathbb{I}_{T^+ \leq T}\) is now growing w.r.t. the threshold \(S_{\text{min}}^*\). Indeed, the higher the threshold (while being less than or equal to \(S_t\)), the higher the probability of crossing. As in the case of growth option, these effects can be partially offset and justify the existence of the interior maximum.

From Figure (9) (with \(S_{\text{min}}^*\) x-axis and \(K_1^-\) y-axis), we notice that there exists an interior maximum in \(S_{\text{min}}^*\) for a given level \(K_1^-\). For the digital case presented here, it is around \(S_{\text{min}}^* = 99\% \times S_0\). Here, a slight decrease in the price of the product sold triggers disinvestment and fire. If the proportional costs increase (remember that here it is in fact the total of actual earnings), then the optimal threshold \(S_{\text{min}}^*\) decreases.

Now we study the second step of the optimization of the equity value. Recall that, according to Equation (28), it is given by

\[E_t = DICc(t, V_{t}^*, D + D_1, T) + DOC(t, V_t, D, V_0^*, T).\]

As shown in Equation (28), the crossing threshold \(S_{\text{min}}^*\) can be obtained by optimal induction of disinvestment and fire (impact of \(DICc\) option) or is equal to \(S_t\) if the \(DOC\) option dominates the total share value of the firm. In this case, the optimal policy is simply to maintain the same investment and employment levels (\(K_1^- = K_0\) and \(L_1^- = L_0\)). Comparison of the two optima depends of course on the choice of various model parameters.

Too high firing costs ( \(q^-\) very high ) discourage such disinvestment and fire. In Figure (10), we analyze the valuation of the share depending on the level of
the original debt $D$. For each level of initial debt, we consider two graphs: the first one corresponds to a large range of values of $S_{\text{min}}^*$; the second one is more concentrated around the "true" threshold $S_{\text{min}}^*$ achieving the optimum (here numerically relatively close to $S_t$).

These graphs show that the equity value is of course a decreasing function according to the value of the original debt $D$. This is due to the fact that this debt corresponds here to an exercise price of a call.

Now we examine the behavior of the option decay which is a potential reduction of both the investment and employment levels. This option is based on the difference between the optimal value of the share obtained by exploiting the possibility of reducing the two levels with the value of the share corresponding to the case where these levels are fixed.

**Proposition 23 (Value of the option decay)**

The value of the option decay $D_o(S_t, K_0, L_0, D)$ is given by: For $t < T_1$,

$$D_o(S_t, K_0, L_0, D) =$$
\begin{equation}
DICc(t, V^*_1, D + D_1, T) + DOC(t, V_t, D, V^*_0, T) - C(t, V_t, D, T).
\end{equation}

In what follows, we illustrate the value of the option decay associated with the possibility of divestment and fire. We consider levels of \( K_1^- \) and \( S^*_\text{min} \) which are not necessarily optimal. In this case, the values of the decay of option may be negative. When the values are positive, the manager prefers to divest and fire.\(^5\) The optimality is obtained of course when the demand reaches the true optimal threshold \( S^*_\text{min} \) and so it defines the optimal capital reduction \( K_1^- \left( S^*_\text{min} \right) \). Note that the optimal reduction in the employment level is given by \( L_1^- = K_1^- \left( S^*_\text{min} \right) \left( \frac{\sigma q}{\beta p} \right)^{-1} \). The option decay is increasing in debt but the monotony with respect to volatility is not verified.

\(^5\)On Figures (10) and (11), the option decay is slightly positive.
Figure 10: Equity value (decay option).
4 Conclusion

This paper examines the problem of investment and hiring (or symmetrically divestment and firing) when demand for the product sold by the company is stochastic and when the manager decisions can be partially reversible. In the framework of Merton (1974) for the modelling of equity and debt values, we provide quasi-explicit values for optimal firm, shares and debt values. It allows us to analyze for example if growth options increase or decrease according to specific financial or management parameters. The results differ quite significantly from those of the standard framework introduced by Tserlukovich (2008). Additionally, in the case of decay option, the optimal solution does not correspond to the abandonment option (whole liquidation). Further extensions could examine the impact of potential debt rescheduling as studied in Moraux and Navatte (2007, 2009) or introduce a convertible debt as proposed in Yagi and Takashima (2012).
References


